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Sixth-Grade Students' Mathematical Abstraction Processes in a Teaching Experiment Designed Based on Hypothetical Learning Trajectory *

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Abstract

This study explored the mathematical abstraction processes of three sixth-grade students who were selected as the focus group in a classroom teaching experiment that was designed based on a hypothetical learning trajectory and determined their abstraction mechanisms with respect to volume measurement in rectangular prisms. The study also determined the role of class social and sociomathematical norms in this process. The teaching experiment was conducted over nine weeks in two stages through whole-class discussions with 12 students and discussions with small groups consisting of three students with low, medium or high academic achievement. One of the small groups was selected as the focus group. The results showed that all the three students in the focus group, particularly the low-attaining student, used reflective (based on thinking) abstraction for volume measurement in rectangular prisms. In this process, it was also observed that social norms such as explanation-justification of ideas and solutions, agreement or disagreement and trying to listen to, understand and question each other and sociomathematical norms such as making an acceptable mathematical explanation-justification and making mathematical solutions played a supporting role in the students' reflective abstraction in addition to their own individual actions.

Keywords

Volume measurement
Hypothetical learning trajectory
Mathematical abstraction
Social and sociomathematical norms
Teaching experiment

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Introduction

The constructivist approach has been the focus of many experimental and theoretical studies in mathematics education. As Simon (1995) suggested, it also contributes to the shaping of reform movements in mathematics education and offers useful ways for mathematics educators to better understand learning and students. On the other hand, although the constructivist approach provides a general framework for how to teach mathematics, it cannot offer a pedagogically specific teaching path towards mathematics teaching (diSessa & Cobb, 2004; Simon, 1995). Therefore, there is a need for instructional models that are based on constructivist approach and sensitive to students' mathematical thinking (Simon, 1995). In other words, focusing on students' mathematical thinking and encouraging mathematical thinking is essential (Simon, 2006). One of the constructivist mathematics teaching tools that take students' thinking into consideration is a theoretical framework called hypothetical learning trajectory [HLT] which was developed by Simon (1995). HLT is seen as an effective tool to understand students' solution strategies, misconceptions and ways learning regarding the teaching of a mathematical concept (Simon, 1995; Simon, et al., 2010).

Although not explicitly stated, current mathematics curricula implemented in Turkey since 2005 were designed with the constructivist approach. Nevertheless, the desired progress in mathematics education has not been achieved yet. Indeed, according to the PISA scores in mathematics literacy by year, the PISA 2015 performance of students in Turkey was lower than their performance in PISA 2009 and 2012, which supports this suggestion (Taş, Arıcı, Ozarkan, & Özgürlük, 2016). Therefore, there seems to be a gap between the implemented curricula and the desired outcomes in two aspects. The first aspect might be the lack of a bridge between learning and teaching as diSessa and Cobb (2004) and Simon (1995) stated while the second aspect might be that the cognitive dimension of learning is more prominent than the social dimension in mathematics teaching in the secondary school mathematics curriculum. Moreover, as Özmantar, Bingölbali, Demir, Sağlam, and Keser (2009) stated, the roles defined for teachers and students in the 2005 curriculum, for example, give clues about the social and sociomathematical norms that should be created in the classroom, but these norms are not directly emphasized. Therefore, in this study, a teaching experiment that addressed volume measurement in rectangular prisms and was designed in accordance with HLT, which is a bridge between learning and teaching, was conducted.

The concept of volume is considered important for geometric concepts such as depth, width, height (dimension), surface, surface area, and space (French, 2004). Therefore, volume measurement is seen as an important part of school mathematics and it is essential that, in addition to their difficulties about this subject, how students develop their understanding of this subject be determined (Kim, 2016). On the other hand, research showed that students had common misconceptions and difficulties in volume measurement (Battista & Clements, 1996; Ben-Chaim, Lappan, & Houang, 1985; Hirstein, 1981; Olkun, 2003; Tan-Şişman & Aksu, 2016; Zembat, 2009). For example, Hirstein (1981) stated that students of different ages made mistakes, such as counting the unit squares and visible unit cubes on surfaces, while calculating the number of unit cubes within prisms-shaped structures and thus they had difficulty in finding the number of unit cubes in these structures. Ben-Chaim et al. (1985) stated that students counted the unit squares and unit cubes on surfaces in prism-shaped structures but they took twice the number they calculated, and that they counted the unit cubes at the edges and corners of the prism more than once and could not visualize the prisms given as drawings. Battista and Clements (1996) stated that, in addition to these general misconceptions and difficulties observed in students, students used the formula "Width x Depth x Height" by heart and they argued that all these situations were caused by insufficient spatial structuring. Studies conducted in Turkey on the subject present results similar to those around the world. For example, Zembat (2009) stated that teacher candidates and students adhered only to the "Width x Depth x Height" formula for volume measurement and, therefore, students reached wrong generalizations by making mistakes in conceptual measurement of volume. Olkun (2003) showed that many students even in the seventh grade had difficulties calculating the number of cubes in structures in the form of a rectangular prism. Tan-Şişman and Aksu (2016) stated that, in addition to these errors, students counted the faces of the cubes in the structures presented visually and they multiplied the result they calculated by three because of the idea that prisms are three-dimensional. The authors also stated that some students used the area measurement formula to measure the volume of prisms, while some other students incorrectly used the volume measurement formulas in different ways.

All these studies revealed that students should be supported about volume measurement. Also, another important reason for conducting this study specifically on this subject was that there were not more recent studies in the literature on this subject. Based on this need, the aim of this study was to determine the abstraction mechanisms of three students selected as the focus group in the experiment class about measuring the volume of rectangular prisms by exploring their mathematical abstraction processes. Cobb and Yackel considered student-student or student-teacher social interaction as an important tool for students' cognitive development (Toluk-Uçar, 2016). There is a strong relationship between the social norms that this interaction is related to and students' conceptual learning (Yackel, Cobb, & Wood, 1993), and norms create learning opportunities (Yackel & Cobb, 1996).

Considering that social and sociomathematical norms, which are included in the social dimension of constructivism, can be an element that supports learning, the study also aimed to address how this supportive role is realized in classroom practices. The research questions that guided the purpose of this study are presented below:

- What are the focus students' abstraction mechanisms at the end of the teaching experiment designed based on HLT framework for volume measurement in rectangular prisms?
- How do the focus students' individual actions and class social and sociomathematical norms affect their mathematical abstraction processes during the teaching experiment designed based on HLT framework?

Theoretical Framework

Aiming to address the cognitive and social perspectives of constructivism in mathematics learning together, this study adopted the Emergent Perspective [EP]. In EP, which blends, coordinates and combines the social and cognitive dimensions of constructivism (Cobb & Yackel, 1996; Mcclain & Cobb, 2001), the coordination of social and cognitive dimensions is based on the idea that the class and individual students cannot be considered separately from each other. Therefore, within the EP theoretical framework, individual actions and interactions of individuals are important for learning (Cobb, 1989, 1990; Cobb, et al., 1991; Cobb & Yackel, 1996; Wood, Cobb, & Yackel, 1995). This led to the idea that rich classroom environments that promote social interactions should be created for qualified mathematical learning (Cobb, Yackel, & Wood, 1992). In the light of this idea, social and sociomathematical norms in the social dimension of EP have attracted attention as an important element in classroom microculture established by teachers and students (Cobb, 1999; Cobb & Yackel, 1996; Yackel & Cobb, 1996). Situations such as explaining and justifying their ideas, trying to understand other student's strategies, agreeing or disagreeing with other students, inquiring other students' solution strategies when misunderstandings occur are considered as social norms (Cobb, Yackel, & Wood, 1989). Mathematical situations such as an acceptable mathematical explanation-justification, a different mathematical solution, a complex solution and an effective solution are expressed as sociomathematical norms (Yackel & Cobb, 1996). Cobb and Yackel (1996) emphasized that social and socio-mathematical norms contribute to the increase in students' mathematical discussions and interaction levels. Class social interaction supports students' mathematical meanings, learning and reasoning (Bauersfeld, 1980, 1988; Cobb, Boufi, McClain, & Whitenack, 1997; Yackel, Cobb, & Wood, 1991). Therefore, there is a strong relationship between norms promoting interaction (Yackel et al., 1993) and students' conceptual learning (Yackel & Cobb, 1996).

Since this study was inherently a class teaching experiment, the study adopted the EP theoretical framework for learning mathematics and, in the light of this theory, the interpretative framework in Figure 1 was created to analyze mathematical learning.

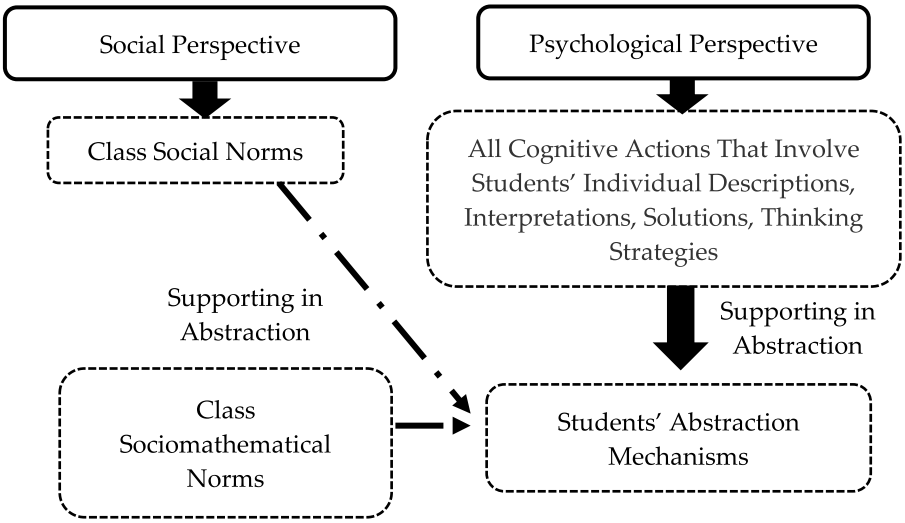


Figure 1. The Interpretative Framework for the Analysis of Mathematical Learning in the Study

As seen in Figure 1, while the students' individual cognitive actions and abstraction mechanisms constitute the psychological perspective of the interpretive framework, the social and sociomathematical norms form the social perspective. In this study, Piaget's abstraction theory was adopted to reveal the abstraction mechanisms of the focus students with respect to the psychological perspective. Piaget argued that learning takes place at different levels in individuals and explained this situation with the abstraction theory, which he introduced to the literature. The abstraction theory developed by Piaget is divided into two main groups: experimental (empirical) abstraction and reflective (based on thinking) abstraction (Cobb, 1994; Piaget, 2001). While reflective abstraction is based on the individual's actions and the mental relationships it establishes regarding the coordination of these actions, experimental abstraction is based on the properties of objects that can be directly observed. Therefore, while the source of reflective abstraction is logical-mathematical knowledge, the source of experimental abstraction is physical knowledge (Von Glasersfeld, 1995; Piaget, 2001; Zembat, 2016). In this regard, this study explored students' abstraction processes related to measuring the volume of rectangular prisms and determined their abstraction mechanisms. In addition to the students' individual cognitive actions that supported their mathematical abstraction, the class social and sociomathematical norms were also taken into account for the analysis of sociological factors. Norms were used in this study to explain how the students supported their mathematical abstractions. In this respect, the study focused on the norms that were accepted and adopted by the students, were frequently observed in the dialogues and led to significant changes in the students' mental actions. All the norms included in this study were used in the way they were developed by Cobb and his team in many studies in line with the EP theoretical framework (Cobb et al., 1989; Cobb & Yackel, 1996; Mcclain & Cobb, 2001; Yackel & Cobb, 1996).

Methods and Research Design

In line with the main purpose of the study, the teaching experiment was used as a research design (Wood, Cobb, & Yackel, 1990). The teaching experiment was designed based on the principles of constructivist approach in accordance with the theoretical framework of the EP.

Hypothetical Learning Trajectory

Simon (1995, p. 135) used the term HLT to refer to "The teacher's prediction as to the path by which learning might proceed". The HLT consists of three main components: the learning objective or goal determined by the teacher for students, the activities or plans made by the teacher to support learning, and the hypotheses of the teacher on how learning will progress. Among other approaches to mathematics teaching, this study adopted the HLT as proposed by Simon (1995). In other words, the HLT was used as a course design tool to provide a pedagogical way to carry out a teaching process compatible with constructivism. In this context, the primary learning objective for the students involved volume measurement in rectangular prisms. For this purpose, the students' proficiency levels and prior knowledge were diagnosed, and the activities and lesson plans were prepared accordingly. The activities about volume measurement in rectangular prisms were prepared based on the hypotheses on how learning would occur. The following hypotheses were formulated on how learning would occur:

- Students can make sense of the volume concept in a way that leads them to reflective abstraction and discover the mental relationships underlying the volume measurement formulas in rectangular prisms.
- Students can construct volume measurement formulas in rectangular prisms by performing reflective (based on thinking) abstraction.

While forming the hypotheses, the researchers took into account the research results suggesting that volume measurement formulas are constructed through memorization; therefore, students need to understand the principles underlying the formulas (Battista, 2007; Battista & Clements, 1996; Zembat, 2009) and they should be encouraged to perform reflective abstraction (Zembat, 2007).

Hypothetical Learning Progression and Teaching Experiment Application Process

The hypothetical learning progression about volume measurement in rectangular prisms is given in Table 1.

Table 1. The Hypothetical Learning Progression for Volume Measurement in Rectangular Prisms

1	Recognize Rectangles and Squares
2	Recognize Rectangular Prisms and Identify Their Basic Properties
3	Develop Counting and Building Skills in Structures Made of Unit Cubes
4	Understand the Necessity of Using Unit Cubes to Make Sense of and Determine the Volume of Rectangular Prisms
5	Understand That the Number of Unit Cubes That Exactly Fill a Rectangular Prism is the Volume of the Rectangular Prism
6	Construct Formulas About Volume Measurement in Rectangular Prisms
7	Use the Formulas About Volume Measurement in Rectangular Prisms in the Context of Daily Life Problems

The teaching experiment implementation process consisted of instructional sequences conducted with the whole class and initial, intermediate and final clinical interviews with the focus students. The instructional sequences lasted nine weeks, two hours a week, and were conducted in three stages. The initial clinical interviews were carried out to identify the students' prior knowledge, as highlighted in HLT teaching framework. In the light of the analysis of data from the initial clinical interviews, a three-week instructional sequence was planned and conducted as the first stage in the HLT in relation to identifying the basic properties of prisms and to recognizing rectangular prisms in order to eliminate any weaknesses in their prior knowledge and help them master the subject. Following this instructional sequence, the first phase of clinical interviews with the focus group students was conducted to understand in more detail how the focus students structure the points discussed in this process, and then a one-week second stage instructional sequence on counting and building skills in structures with unit cubes was planned and implemented. After the second stage instructional sequence, the second phase of clinical interviews with the focus group students were conducted, and then a five-week third stage instructional sequence involving the HLT on volume measurement in rectangular prisms, the primary learning objective in the study, was planned and implemented. This was followed by the final clinical interviews with the focus group students that were conducted in order to elaborately examine how the students constructed volume and volume measurement in rectangular prisms and to identify the mechanisms that demonstrated each student's mathematical abstraction in relation to volume measurement in rectangular prisms.

In this study, one of the researchers played a role as both a researcher and a teacher. The researcher teacher led the class practices during the teaching lessons and guided the students. The other researcher, along with the researcher teacher, played an active role in designing activities, planning teaching, creating data collection tools and data analysis, and took part in practices as an observer.

Teaching Activities and Materials Prepared in the Light of Theoretical Framework

In this process, a learning progress was specified within the framework of HLT, which is seen in Table 1. In accordance with this progress, lesson plans and activities were prepared in relation to the weekly subjects in the teaching lessons with an expert mathematics educator. The activities were prepared based on the research results in the literature (Battista & Clements, 1996; Ben-Chaim et al., 1985; Hirstein, 1981; Olkun, 2003; Tan-Şişman & Aksu, 2016; Zembat, 2009). In this way, the activities were designed to enable students to discover the mental relationships underlying the volume measurement formulas in rectangular prisms, and to encourage students to think. Therefore, the activities of forming volume formulas in rectangular prisms were designed as rectangular prism representations, where all unit cubes were built first, then only its dimensions were built, and then the surfaces were unit squares. At the same time, the activities were designed through contexts students were usually familiar with in everyday life. The designed teaching activities are presented in Table 2. Within the framework of the social perspective of the study, the instructional sequences were performed in two stages: small group and class discussions. Meanwhile, the teaching process was carried out in a way that would allow small groups to interact with each other and then as class discussions about the results achieved by the groups and the paths they followed.

Classroom Micro-culture

The teaching sessions were conducted by one of the researchers. Having worked as a teacher and taught the participants since the fifth grade, the researcher knew his students well. In addition, he was aware of the role of norms in supporting learning as he kept up to date with the recent literature in mathematics education. He takes norms into account when conducting mathematics lessons. Therefore, in class implementations in general, before the teaching sessions, the participants had already demonstrated some behaviors such as explaining and justifying mathematical solution processes, listening to their friends, and inquiring and respecting each other. In other words, the participants were already familiar with a classroom culture created within the context of the norms addressed in this study. The teacher asked the students to exhibit these norms that they were used to in small group discussions during the teaching process and encouraged the use of these norms in class discussions. The teacher himself exhibited these behaviors, too.

Table 2. Teaching Activities and Materials

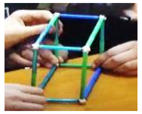
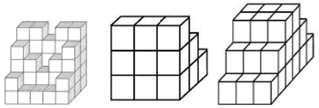
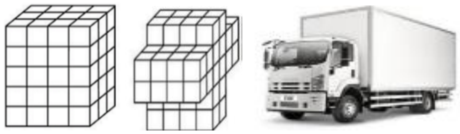
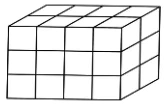
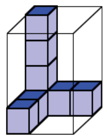
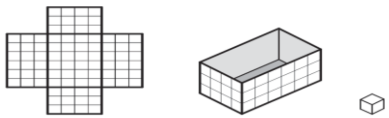
Week	Objective	Activities Designed and Materials Used	Embedded Mathematical Practices	Sample Discussion Questions
1	Recognize rectangles and squares (Recall prior knowledge of two dimensions before moving to three dimensions)	**Activity**: A Football Field 	• Identifying the length and width of a rectangle and square • Recognizing rectangular and square surfaces • Calculating the surface area of a rectangle and square	1. What can we call the distance between the two goals of this football field? 2. What can we call the surface of a football field covered with grass without any leaving space?
2 and 3	Recognize rectangular prisms	**Activity**: A Refrigerator **Materials:** Daily Life Models and Geomag Bars   	• Identifying the basic components of rectangular prisms • Identifying the dimensions and bases of rectangular prisms	1. If our refrigerator had the shape of a square prism, what would the surfaces of our refrigerator look like? 2. If our refrigerator had the shape of a square prism, would it be clear which surfaces were the base surfaces of our refrigerator? Why?
4	Develop counting and building skills in unit cube structures and make sense of volume	**Activity:** Unit Cube Structures with Samples Given Below  **Material:** Unit Cube Sets   	• Computing the number of unit cubes in structures made of unit cubes and building these structures using unit cubes • Building rectangular prisms with unit cubes and calculating the number of unit cubes • Emphasizing that the space covered by the structures made of unit cubes is the volume	1. Can you find the number of unit cubes in each structure? 2. Using unit cubes, can you build a rectangular prism that measures 3 units width, 2 units depth, and 4 units high?
5	Make sense of the volume of rectangular prisms	**Activity:** A Rectangular Prism Box  **Materials:** A Rectangular Prism Box and Visual Representations of Prism and Globe from Daily Life	• Understanding the necessity of using unit cubes to calculate the volume of rectangular prisms	1. Can you measure the exact volume of the box when the box is filled with the tennis balls below? Please explain why. 2. Can you measure the exact volume of the box when the box is filled with any liquid? Please explain why.

Table 2. Continued

Week	Objective	Activities Designed and Materials Used	Embedded Mathematical Practices	Sample Discussion Questions
6	Understand that the number of unit cubes that exactly fill a rectangular prism is the volume of rectangular prisms	**Activity:** Soap bars and Truck Trailer  **Materials:** Unit Cube Sets	• Developing the skills of calculating the number of unit cubes in rectangular prisms and other structures made of unit cubes	1. How many pieces of soap bars in the form of unit cubes are there in the structures? Please discuss how you calculated. 2. Given that all the unit cubes of soap bars in the structures completely fill the truck trailer, how many unit cubes of soap is the volume of the truck trailer?
7, 8 and 9	Construct volume measurement formulas in rectangular prisms and use them in the context of daily life problems	**Activities**: 1) A rectangular prism built with soap bars in the form of unit cubes  2) A square prism-shaped box with dimensions built with unit cubes  3) Placing unit cubes into a box whose surfaces are covered with unit squares  4) Problems prepared in relation to daily life contexts about volume measurement in rectangular prisms **Materials:** Unit Cube Sets, Use of Technology	✓ Discovering different strategies in calculating the number of unit cubes in rectangular prisms made of unit cubes	1) In Activity 1, what is the number of soap bars in the form of unit cubes? Please explain how you calculated it. 2) In Activity 2, how many more unit cubes are needed to completely fill the box? 3) In Activities 1 and 2, do you think there are different ways to calculate the volume of prisms? Please discuss. 4) In Activity 3, can there be some shortcuts to calculate the number of unit cubes that the box can take without placing the unit cubes in the box? Please discuss. 5) Half of a container in the form of a square prism with a surface area of the bases 25 br^2 and a height of 8 br is filled with oil. Please calculate the volume of oil in the container?

Participants

A total of 12 sixth-grade students at a public school voluntarily participated in the study. The school where the study was conducted was located in a medium-to-low socio-economic area. The small groups in instructional sequences were created by taking into account students with low, medium and high level of academic achievement so that each of the groups could be as heterogeneous as possible and those who might study together in harmony and communicate with each other more easily. One of these groups was designated as the focus group, and in this group, the student with low achievement was given the pseudonym Ali, the student with medium achievement was given the pseudonym Emre, and the student with high achievement was given the pseudonym Murat. While presenting the findings, the initials of the names of Ali, Emre and Murat were used in the clinical interview tables.

Data Collection

Data were collected through the initial, intermediate and final clinical interviews with the focus group students, video recordings of teaching sessions and small group activities, worksheets used in small group activities and student journals. With each of the focus students, the researcher-teacher carried out and video-recorded four clinical interviews each of which lasted approximately one class hour: one at the beginning of the research, two during the teaching sessions and one at the end of teaching. In teaching sessions, the small groups studied on the tasks first and then class discussions were held about the results obtained by these groups and the way they obtained those results. In the initial clinical interviews, the focus group students were asked questions about recognizing rectangular prisms; determining basic component properties such as base, surface and dimensions; calculating the number of unit cubes in different prismatic or non-prismatic structures created with unit cubes; and building structures whose visual representations are given with unit cubes. The first and second intermediate clinical interviews were conducted to understand how the weaknesses identified in the initial clinical interviews changed over the four-week period. In the final clinical interviews, the students were asked questions to understand how they used the unit cubes to measure the volume of rectangular prisms and how they constructed the formulas they formed as a result. In addition, in the study, at the end of each lesson, all the students were given journals with semi-structured questions about what happened in that lesson. The journals were prepared in a semi-structured way, considering that the students might have difficulty expressing their feelings and thoughts considering their grade level. The semi-structured journals were prepared to explore the points that the students learned or had difficulties during the lesson, their harmony with each other in small group activities, and their feelings and thoughts about video-recording in the classroom.

Data Analysis

Data were analyzed in two phases: ongoing analysis and retrospective analysis. Table 3 shows the stages of ongoing analysis and retrospective analysis used in this study.

Table 3. Data Analysis Process

Stage 1		**Stage 2**		**Stage 3**	
After the Initial Clinical Interviews	Ongoing Analysis 1	After the Instructional Sequences and Intermediate Clinical Interviews	Ongoing Analysis 2	After the Final Clinical Interviews	Ongoing Analysis 3
Retrospective Analysis					
Abstraction Mechanisms of Students					

In the ongoing analysis phase of the study, the researchers independently conducted macro analyses firstly by watching each clinical interview and the videos recorded at the end of each lesson and examining the student worksheets and diaries. Afterwards, the researchers came together to discuss the results and determine the difficulties the students had and the points the students constructed, and the researchers designed the weekly teaching lesson plans and activity contents.

In the retrospective analysis of the research, all the clinical interviews and teaching sessions conducted with the focus students were transcribed. First, the researchers independently conducted microanalyses on the videos of each clinical interview and teaching sessions, and the data set obtained from the worksheets used in the focus group discussions were subjected to microanalysis by the researchers independently of each other. In this process, as a result of the analyses they conducted independently, the researchers determined the themes, sub-themes and codes related to the "physical/mental actions" demonstrated by the students in each clinical interview with the focus group students. As a result of the analyses of the instructional sequences, the researchers determined the main themes of "physical-mental actions that the students had difficulty in or constructed" and "class norms" about "the actions they exhibited" during the weekly small group and class discussions as well as the sub-themes and codes for these themes. The two researchers then discussed the results of the analysis on the whole data set, taking into account the independent analyses of each other, and revised the themes, sub-themes and codes of the clinical interviews and teaching sessions. In order to ensure reliability, the codes of the two researchers were compared and a consensus was reached by discussing the differences of opinion on each code. As a result of all the analyses performed, the two researchers identified the abstraction mechanisms of Ali, Emre and Murat about volume measurement in rectangular prisms.

Results

The results are presented under three headings: initial clinical interviews, first and second stage instructional sequences and intermediate clinical interviews, and third stage instructional sequences and final clinical interviews.

Results from the Initial Clinical Interviews

The findings related to the physical-mental actions exhibited by the students in the initial clinical interviews are presented in Table 4 under two main themes: recognizing rectangular prisms and determining their basic properties, counting and building structures with unit cubes.

As can be seen in Table 4, while Emre, the student with medium academic achievement, and Murat, the one with high achievement, perceived rectangular prisms in three dimensions and named them correctly, Ali, who had low achievement level, mistook the rectangle for the rectangular prism and the square prism and the square for the cube. At the same time, Ali made a formal analogy by matching the visual representations of the rectangular prism with its concrete representations and said, "They are similar to each other". While Ali could not make any comments about the unit cube, which has an important place in constructing the concept of volume and its formulas, Emre described the unit cube as "a little thing" and Murat described it as "an object not different from any regular cube".

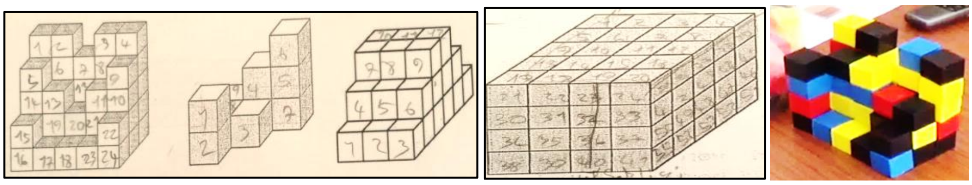
While explaining the basic properties of rectangular prisms, the students had difficulty particularly in recognizing the base and dimensions of prisms. Ali and Emre stated that rectangular prisms had one base and considered the base as "the part of the prism touching the ground". However, none of the three students were able determine the dimensions of rectangular prisms accurately. The students especially mistook "the width" dimension for "the height".

Table 4. The Physical-Mental Actions Exhibited by the Focus Students in the Initial Clinical Interviews

Recognizing Rectangular Prisms and Determining Their Basic Properties		**Counting and Building Structures with Unit Cubes**	
✓ *Perceiving rectangular prisms* -Three-dimensional (E, M) - Two-dimensional (A) -Formal similarity (A) *Perceiving unit cubes* -Formal (E) -Edge discontinuity (M)	✓ *Concept of base* -Face touching the ground (A, E) ✓ *Confusion about dimensions* (A, E, M)	✓ *Calculating the number of unit cubes* - Counting visible cubes one by one (A, E, M) - Counting visible cubes one by one (A) - Multiple counting of corner and edge cubes (A) - Counting cubes mentally (E) -Using layering and ordering strategies (E, M)	✓ *Building structures* -Making the shape look similar to the one in the visual (A,E,M) - Using ordering strategy (A, E, M) - Using layering strategy (E)

The students were presented with the visual representations of a square prism made of unit cubes and three different structures that were not prisms. The students were asked to calculate the number of unit cubes making up the structures in the visual representation first and then on the concrete representation by using the unit cubes. As seen in Table 4, the students used appropriate strategies, even partially, in counting and building structures. On the other hand, the students mostly used inappropriate strategies and, therefore, they experienced various difficulties in both calculating the number of unit cubes on visual representation and creating a concrete representation and calculating the number of cubes on this representation. These difficulties varied depending on the structures given.

As seen in Figure 2, Ali calculated the number of unit cubes in the visual representation by writing on the faces of the cubes he saw in the structures that were not prisms and by counting the surfaces he saw in the square prism one by one. Ali also counted the corner and edge unit cubes in the square prism more than once.



(a) (b) (c) (d) (e)

Figure 2. Ali's Actions to Calculate the Number of Unit Cubes on Visual Representation During the Initial Clinical Interview and an Example of His Concrete Representation

When building the structures with unit cubes, Ali created one of the visual representations (Figure 2c) using the ordering strategy (front-to-back ordering) and he built the others by imitating the formal appearance of the visual representation. While calculating the number of unit cubes in the structures that he built, he counted the unit cubes that he saw one by one. However, as can be seen in the figure (Figure 2e), Ali inaccurately built the structures in general and even miscalculated the number of unit cubes in the structures that he built inaccurately.

Although Emre was aware of the nonvisible unit cubes in the visual representation of the structures, he failed to calculate some unit cubes in Figure 2a. While he made the calculations for Figures 2b and 2d accurately, he was just able to find 36 in Figure 2c, where the number of cubes varied between

30 and 38. As can be seen in Table 4, while performing these calculations, Emre tried to mentally count the visible and nonvisible unit cubes from front to back in Figure 2a saying, "There are 7 in front, 7 behind them, so there are 40 if we count the visible ones, too". While making these calculations, Emre counted the unit cubes in groups by using the front-to-back ordering strategy in Figure 2a, counting one by one in Figure 2b, using the layering strategy (calculating each layer separately) in Figure 2c, and using the ordering strategy (calculating with front-to-back ordering) in Figure 2d. While constructing the concrete representations of structures in the visual representation, Emre constructed Figures 2b and 2d accurately, but he constructed Figure 2a inaccurately. In Figure 2c, on the other hand, he came up with only one of the accurate constructions of the structure in the same way as he thought about the calculation in the visual representation. Emre built Figures 2a and 2b by making them look similar to the visual representation view, he built Figure 2c by using the front-to-back ordering strategy, and he built Figure 2d using the layering strategy.

Murat miscalculated the number of the unit cubes in the visual representation of the structures in Figure 2a because he counted only the visible cubes. Despite making accurate calculations in Figures 2b and 2d, he followed a method similar to Emre's strategy to calculate the number of unit cubes in Figure 2c. In addition, for these calculations, except for the structure in Figure 2a, Murat used the same strategies as used by Emre. While constructing the concrete representations of the structures in the visual representation, Murat accurately built all the structures except for the one in Figure 2c. In Figure 2c, on the other hand, he came up with only one of the accurate constructions of the structure in the same way as Emre. While Murat formed the square prism and the third one of the different structures in order, he created the first and the second ones by comparing it to the visual representation.

Results from the First and Second Stage Instructional Sequences and Intermediate Clinical Interviews

The first and second stage instructional sequences, which lasted four weeks, were planned by taking into account the points about which the students were observed to have incomplete prior knowledge, difficulties or misconceptions during the initial clinical interviews with the focus group students.

The First Stage Instructional Sequence

The actions exhibited by the students in the first stage instructional sequence are presented in Table 5, under the main themes of "physical-mental actions that the students had difficulty in and constructed" and "class norms".

The objective in the first week activity was to recognize rectangles and squares. For this activity, a scenario was prepared on a football field context, which the students were already familiar with in daily life. During this activity, as seen in Table 5, the students generally associated rectangle and square with daily life objects in the small group and class discussions. Especially in the small group discussions, Ali and Emre, two of the focus group students, gave objects such as doors and windows in the daily life as examples of a rectangle and a square, while Murat gave the surfaces of these objects as examples. During the group discussion, Murat *disagreed with* Ali and Emre's statements saying, "They are not" and gave a persuasive *mathematical explanation* saying, "The door is not all rectangular, but the face of the door is rectangular". Therefore, the other two students' thoughts shifted from the object to the surface of the object thanks to Murat's explanations. In order to overcome these difficulties observed in the classroom practices, the teacher pointed out the differences between rectangular and square objects and daily life objects by using object representations that are frequently used in daily life.

Table 5. The Actions Exhibited by the Focus Students in the First Stage Instructional Sequence

Week I	Week II	Week III
Actions That the Students Had Difficulty in and Constructed		
✓ *Identifying the rectangle and square with objects* [Small Group Discussion (**SGD**); Class Discussion (**CD**)]	✓ *Perceiving rectangular prisms* -Formal similarity ✓ *Concept of Base* -Face touching the ground ✓ *Confusion about dimensions* ✓ *Perceiving unit cubes* -Formal **[SGD, CD]**	✓ *Being able to build rectangular prisms* ✓ *Being able to determine the basic properties of rectangular prisms* **[SGD, CD]**
Class Norms		
✓ *Social Norms* -Explaining ideas -Disagreeing or agreeing with explanations -Trying to listen to and understand each other ✓ *Sociomathematical Norms* - Making mathematical explanations	✓ *Social Norms* -Inquiring each other -Explaining and justifying ideas -Disagreeing or agreeing with explanations -Trying to listen to and understand each other ✓ *Sociomathematical Norms* -Making acceptable and different mathematical explanations/justifications	✓ *Social Norms* -Inquiring each other -Explaining and justifying ideas -Disagreeing or agreeing with explanations -Trying to listen to and understand each other ✓ *Sociomathematical Norms* - Inquiring each other -Explaining and justifying ideas -Disagreeing or agreeing with explanations -Trying to listen to and understand each other

In the second week activity, in which the objective was to recognize rectangular prisms and identify basic properties of prisms, a scenario was prepared on the refrigerator context, which the students were already familiar with in daily life. In addition, rectangular prism models frequently used in daily life were also used. In the small group discussion, Ali matched the visual and concrete representations of the rectangular prism according to their formal similarities to each other, but in the group discussion, he changed this erroneous idea with Emre and Murat's *acceptable mathematical explanations* based on the properties of prisms. On the other hand, Ali's confusion about dimensions, which was previously observed, was again observed in the group discussion. In the meantime, Ali was able to perceive the dimensions of the prism as a result of Emre and Murat's *inquiries* such as "... Can you show the width, depth and height of this prism?" In particular, Murat moved on the dimensions of a rectangle and made a *mathematical explanation* that the height dimension is different in prisms from rectangles. As can be seen in Table 5, while Ali and Emre defined the base of rectangular prisms as a single surface towards the ground, Murat was able to show the base in every case by stating that the base was two surfaces. However, although Murat showed the base surfaces correctly, Ali and Emre could not construct the base surfaces correctly because Murat could not give a convincing answer to Emre's *inquiring*, "In a square prism, the square surfaces are the bases no matter how we place it. However, they are always at the bottom in a rectangular prism and a cube and you say top surfaces are also bases. Why?" In addition to this, during the small group discussions, the three students perceived the unit cube as something "small" in terms of shape and could not make any correct explanations except that the surfaces of unit cubes were square. Similarly, in class discussions, in general, the students had difficulty in determining the dimensions and base surfaces of rectangular prisms and recognizing unit cubes. The teacher used the dimensions of the rectangle in determining the dimensions to overcome the difficulties in the classroom practices, and explained the condition required for the base in identifying the base surfaces. For the unit cubes, the teacher made use of the term "unit cube", which indicated that each dimension was a unit. In this process, as a result of the teacher's *explanations* and *inquiring* the students and the support of the norms such as the students' *inquiring, disagreeing, justifying* and *explaining* their thoughts, difficulties of students, especially those of Ali and Emre, were overcome, and the students came to an *agreement based on the discussions*.

In the third week activity, geomag bars and magnets were used as teaching materials to further improve and reinforce the perception of three-dimensionality. In the small group and class discussions, the students generally used geomag bars and magnets to build rectangular prisms, and they were able to identify the basic properties in the prisms they built. In addition, social norms such as *explaining-justifying their ideas, trying to understand the explanations of their friends, agreeing,* and *listening to and inquiring each other*, and socio-mathematical norms such as *making an acceptable and different mathematical explanation-justification* were often observed during the small group and class discussions in this activity. Especially Ali was more confident and participative within the group in that week's activity compared to the first two weeks, he was very willing to answer questions of Emre and Murat, and he said, "I can answer if you have any questions".

The First Intermediate Clinical Interviews

The findings related to the physical-mental actions exhibited by the students during the first intermediate clinical interviews, which were performed after the three-week first instructional sequence, are presented in Table 6 under the main theme of "recognizing rectangular prisms and identifying their basic properties".

Table 6. The Physical-Mental Actions Exhibited by the Focus Students During the First Intermediate Clinical Interviews

Recognizing Rectangular Prisms and Identifying Their Basic Properties	
✓ *Perceiving rectangular prisms* -Three-dimensional thinking (A, E, M) -Perception based on basic properties (A, E, M)	✓ *Concepts of base and dimension* -Being able identify base surfaces (A, E, M) -Being able identify dimensions (A, E, M)

As seen in Table 6, in the first intermediate clinical interviews, the three focus group students were observed to come up with constructions about recognizing rectangular prisms and identifying the basic properties of prisms in a way different from they followed during the initial clinical interviews. In contrast to what he did during the initial clinical interview, Ali perceived rectangular prisms as three-dimensional and considered the basic properties of prisms while matching visual and concrete representations of rectangular prisms. The students stated that each dimension of a unit cube is a unit and all the faces are a unit square. However, contrary to the case in the initial clinical interviews, Ali and Emre were able to accurately determine the base surfaces of rectangular prisms in all cases, and all the three students were able to make appropriate justifications in determining the base surfaces. In addition, all the three students were able to accurately determine the size of the rectangular prisms in each case.

The Second Stage Instructional Sequence

The actions exhibited by the students in the second stage instructional sequence are presented in Table 7, under the main themes of "physical-mental actions that the students had difficulty in and constructed" and "class norms".

Table 7. The Actions Exhibited by the Focus Students in the Second Stage Instructional Sequence

Week IV	
Actions That the Students Had Difficulty in and Constructed	
✓ *Counting and Building Structures with Unit Cubes*	
-Counting visible unit cubes one by one -Counting unit cubes mentally -Using layering and ordering strategies -Width x Depth x Height **[SGD, CD]**	-Calculating by considering nonvisible unit cubes -Using layering and ordering strategies **[CD]**
Class Norms	
✓ *Social Norms*	✓ *Sociomathematical Norms*
-Inquiring each other -Explaining-justifying ideas and solutions -Disagreeing or agreeing with explanations -Trying to listen to and understand each other	-Making acceptable and different mathematical explanations/justifications -Performing easy and effective mathematical solutions

In the fourth week activity, in which the objective was to improve counting and building skills, a scenario was prepared on unit cubes that students were already familiar with in their daily life to have them engage in activities of performing calculations on different structures made of unit cubes and building concrete representations of structures. As can be seen in Table 7, in the small group discussion, while calculating the number of unit cubes in Figure 2a, in which the focus group students had difficulty during the initial clinical interviews, Ali counted the cubes he saw in the visual representation one by one and Emre mentally counted the visible and nonvisible unit cubes starting from the front to the back. On the other hand, Murat, unlike what he did during the initial clinical interview, started counting from the bottom and calculated the number of unit cubes by using layering and ordering strategies. It was observed that the classroom norms mentioned in Table 7 guided the mental actions of the students in the actions that took place during the small group discussions. For example, Emre *disagreed with* Ali's strategy stating that Ali did not count the nonvisible cubes and let Ali recognize the cubes that were not seen directly from that perspective. Nevertheless, Murat's *easy and effective mathematical solution*, which led to the result more easily, created awareness in the other students and this strategy (layering and ordering) was adopted by the other two students and enabled them to use it in different structures. Moreover, while building a rectangular prism as a group using unit cubes, they first constructed the dimensions and then completed the remaining parts. They also used the "Width x Depth x Height" strategy when calculating the number of unit cubes in the rectangular prism they built, as seen in Table 7.

During the class discussions, as seen in Table 7, the students generally had difficulty in calculating the number of units in non-prismatic structures made of unit cubes in the visual figure and building the concrete representations of structures with unit cubes, and they reflected these situations, which they had difficulty in, in their journals. The teacher carried out several classroom activities on building structures by using unit cubes so that the students could recognize nonvisible unit cubes and discover different strategies such as layering and ordering to calculate the number of unit cubes on visual and concrete representations. While the students generally showed signs of overcoming these challenges in the class discussions, several students were not able to overcome the difficulties they surfaced in the structures that could have different formations. In the class discussion process, the teacher's *explanations and inquiry*, *explanations made by students*, especially by Murat, about their *solution strategies* for calculating the number of unit cubes, and *inquiry* of these strategies by other students asking, "How did that happen? Why did it happen so?" helped the students discover and adopt strategies such as layering and ordering, and overcome their difficulties.

The Second Intermediate Clinical Interviews

The findings related to the physical-mental actions exhibited by the students during the second intermediate clinical interviews conducted after the one-week second stage instructional sequence, are presented in Table 8 under the main theme of "counting and building structures with unit cubes".

Table 8. The Physical-Mental Actions Exhibited by the Focus Students During the Second Intermediate Clinical Interviews

Counting and Building Structures with Unit Cubes	
✓ *Calculating the number of unit cubes* -Taking nonvisible unit cubes into consideration (A, E, M) -Using layering and ordering strategies (A, E, M) -Multiplicative reasoning (A, E, M)	✓ *Building structures* -Using layering and ordering strategies (A, E, M)

In the second intermediate clinical interviews, all the three students were observed to construct the subjects of counting and building structures made of unit cubes in a way different from they followed during the initial clinical interviews. In contrast to what he did in the initial clinical interview, explaining, "If there were no cubes in the back and at the bottom, those on the top would fall", Ali took the nonvisible unit cubes into consideration, he did not count the unit cubes more than once and he did not consider only the visible surfaces of the structures while calculating the number of the unit cubes in structures made of unit cubes, as seen in Table 8. Also, when calculating the number of unit cubes in each layer in structures made of unit cubes, Ali used multiplicative reasoning and said, "Here we have 5, 3 rows of 5, so it is 3x5 = 15 units of cube". In addition, as seen in Table 8, unlike what they did during the initial clinical interviews, all the three students used layering and ordering strategies in a way that was more meaningful and consistent with the structure among all the different structures while counting and building structures made of unit cubes. Finally, they were able to construct concrete representations of the structures made of unit cubes by using layering and ordering strategies and to calculate the number of unit cubes in these structures by using the same strategies, as seen in Table 8.

Results from the Third Stage Instructional Sequences and Final Clinical Interviews

The five-week third stage teaching series activities were planned and implemented based on the hypothesis that students can create volume measurement formulas themselves and perform reflective abstractions considering the internal construction process, and the final clinical interviews were held with the focus students at the end of the process.

The Third Stage Instructional Sequence

The actions exhibited by the students in the third stage instructional sequence are presented in Table 9, under the main themes of "physical-mental actions that the students had difficulty in and constructed" and "class norms".

Table 9. The Actions Exhibited by the Students in the Third Stage Instructional Sequence

Week V	Week VI	Week VII	Week VIII	Week IX
Physical-Mental Actions That the Students Had Difficulty in and Constructed				
✓ *Making sense of the volume of rectangular prisms* -Being unable to determine the volume by filling a rectangular prism with unit cubes **[SGD]** -Being unable to determine the volume if a rectangular prism is filled with different prisms -Being unable to integrate the volume formula into the volume of liquids **[SGD, CD]**	✓ *Volume of a prism filled with unit cubes* -Being able to relate the volume of a prism filled with unit cubes with the number of unit cubes -Using layering and ordering strategies **[SGD, CD]**	✓ *Developing volume measurement formulas* -Discovering the Width x Depth x Height volume measurement formula and its variations **[SGD, CD]** -Being unable to move from three-dimension to two-dimension **[CD]**	✓ *Developing volume measurement formulas* - Discovering the Width x Depth x Height volume measurement formula and its variations **[SGD, CD]** - Being unable to move from three-dimension to two-dimension **[SGD]** - Being able to move from three-dimension to two-dimension **[CD]**	✓ *Using volume measurement formulas in the context of daily life problems* -Constructing Base area x Height formula along with other volume measurement formulas **[SGD, CD]**
Class Norms				
✓ *Social Norms* -Inquiring each other -Explaining and justifying ideas -Trying to listen to and understand each other -Agreeing with explanations ✓ *Sociomathematical Norms* -Making acceptable and different mathematical explanations-justifications	✓ *Social norms* -Explaining and justifying ideas -Trying to listen to and understand each other -Agreeing with explanations ✓ *Sociomathematical Norms* -Making acceptable mathematical explanations-justifications -Performing effective solutions	✓ *Social norms* - Explaining and justifying ideas and solutions -Trying to listen to and understand each other -Agreeing with explanations ✓ *Sociomathematical Norms* -Making acceptable mathematical explanations-justifications -Performing easy, effective and different solutions	✓ *Social norms* -Inquiring each other -Explaining and justifying ideas -Disagreeing or agreeing with explanations -Trying to listen to and understand each other ✓ *Sociomathematical Norms* -Making acceptable and different mathematical explanations-justifications	✓ *Social norms* -Inquiring each other -Explaining and justifying ideas -Disagreeing or agreeing with explanations -Trying to listen to and understand each other ✓ *Sociomathematical Norms* -Making acceptable and different mathematical explanations-justifications -Performing effective solutions

For the fifth week activity, in which the objective was to realize the necessity of using unit cubes to make sense and calculate the exact volume of rectangular prisms, a scenario was prepared based on the carrying capacity of an empty box. As can be seen in Table 9, during the small group discussions, Ali had difficulty in understanding the necessity of using a unit cubes to make sense of and determine the exact volume of rectangular prisms. Similarly, during the class discussions, some other students had difficulty understanding that the volume of the rectangular prism could not be determined precisely if it was filled with different prisms other than unit cubes. In order to overcome these difficulties in classroom practices, the teacher gave the representation of a rectangular prism used in everyday life as an example and tried to ensure that the students could realize the case here by initiating a discussion about whether a rectangular prism would always be fully filled with different prisms. At the end of the practices, the students generally overcame these difficulties and *came to an agreement* based on the discussions. Meanwhile, several students, especially Murat, showed on concrete representation that if a prism were filled with unit cubes, there would be no gap in the prism, so that the volume of the prism could be determined. These students expressed their thoughts with *acceptable mathematical explanations and justifications* and other students *inquired* them in the classroom, which contributed to overcoming difficulties. On the other hand, as seen in Table 9, during the small group and class discussions, the focus students made a mathematical explanation saying, "When a rectangular prism is filled with liquid, the volume of the prism cannot be calculated if the volume of the liquid is unknown." The students thought that the "Width x Depth x Height" volume measurement formula, which they discovered in the fourth week, was valid only in unit cube structures and could not transfer the formula here.

In the sixth week activity, in which the objective was to understand that the number of unit cubes placed in a rectangular prism without leaving any space in it was the volume of the rectangular prism, a scenario was prepared upon placing soap bars in the shape of unit cubes, which the students were already familiar with in daily life, in a rectangular prism-shaped truck trailer. In this activity, the students did not have any difficulties in the small group and class discussions and, in general, it was once again observed that they constructed the calculation of the number of unit cubes in a rectangular prism and different structures made of unit cubes, as seen in Table 9. The students *agreed* upon layering and ordering strategies that offer *an effective solution* to calculate the number of unit cubes.

In the seventh and eighth week activities, the objective was to construct volume measurement formulas in rectangular prisms. For the seventh week activity, a scenario was prepared based on a rectangular prism built with soap bars in the form of unit cubes, which the students were familiar in daily life. During the small group and class discussions, the students first calculated the volume of the rectangular prism made of unit cubes by using layering and ordering strategies without using the volume measurement formula as seen in Figure 3. Then the students were asked to discover volume measurement formulas for rectangular prisms by using concrete unit cubes and the knowledge they constructed. As seen in Table 9, during the small group discussion in the activity, Emre was the first student to construct the "Width x Depth x Height" volume measurement formula. About this subject, Emre suggested *an acceptable mathematical explanation and a solution that his friends found easy* by saying, "There is an easy way to calculate the number of soap bars. There are 2 soap bars in a row and 4 rows. 4 times 2 is 8. There are 8 in the first layer, and because the height is 3, we multiply 8 by 3. And we get 24." Murat, on the other hand, constructed a different volume measurement formula: "Height x Depth x Width". However, while constructing this formula, he used the strategy of counting unit cubes by separating them from the side as seen in Figure 3. After discovering these formulas, Emre counted the unit cubes by ordering them starting from the front to the back and discovered the "Height x Width x Depth" formula. It was observed that the *mathematical explanations and solutions* made by Murat created an association in Emre's mind and helped Emre discover this formula in this way. In fact, after Murat's discovery of the "Height x Depth x Width" formula, Emre also discovered this formula and made *an acceptable mathematical explanation and solution* by saying, "Then we first multiply the height by the width: 4 times 3 is 12 and then we multiply the result by the depth: 12 times 2 is 24. This is OK, too." The focus students realized that, in these formulas, multiplying the width by the depth gave the number of unit cubes in the first layer, multiplying the depth by the height gave the number of unit cubes in the last row of the side, and multiplying the height by the width gave the number of unit cubes in the front.

Figure 3. The Students' Actions of Discovering Volume Measurement Formulas on The Rectangular Prism Made of Unit Cubes in The Seventh Week Small Group Discussions

Similarly, during the class discussions in general, the students discovered the "Width x Depth x Height" volume measurement formula and other formulas created by multiplying the dimensions in different orders in this formula. On the other hand, the students generally associated the dimensions with the number of unit cubes while constructing the formulas. In the process, the students switched from three dimensions (e.g. "width is 4 unit cubes") to one dimension (e.g. "length is 4 units") and constructed the formula more accurately. Meanwhile, the students in another group in the class constructed a different volume measurement formula as "Width x Depth = Area of the first layer and Area of the first layer x Height" as seen in Figure 4. The students of that group associated surface area of the base with the number of cubes in the first layer rather than the unit squares in the base surface. This thought was also observed with other students, except for Murat and few other students. Therefore, in the class discussions in general, the students switched from three dimensions to two dimensions, and they had difficulty in accurately constructing the "Surface area of the base x Height" volume measurement formula. As seen in Figure 4, although the teacher used the visual and concrete representations of the rectangular prism made of unit cubes so that the students could overcome this difficulty, it could not be overcome.

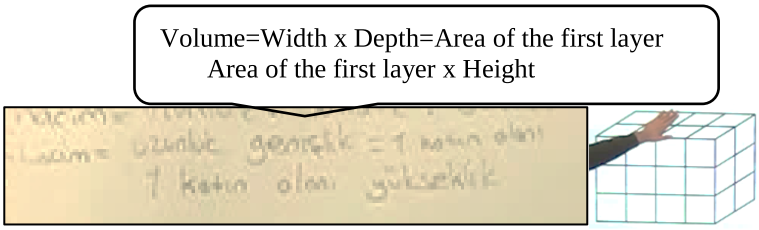


Figure 4. The Volume Formula That A Group Discovered in The Class Discussion and The Teacher's Actions Towards the Difficulties Experienced In This Formula

For the same objective, in the eighth week activities, two scenarios were prepared based on boxes in the shape of a square prism with dimensions made of unit cubes and an empty rectangular prism with surfaces covered with unit squares. In the small group discussions, the focus group students constructed the "Width x Depth x Height" formula, the different forms of this formula, and the "Number of unit cubes in the first layer x Height" volume measurement formula. On the other hand, the focus group students except for Murat had difficulty in constructing the "Surface area of the base x Height" formula, or in switching from three dimensions to two dimensions in other words. In the "Surface area of the base x Height" volume measurement formula, Ali and Emre associated the surface area of the base with the number of cubes in the first layer and reflected this idea on the worksheet as shown in Figure 5.

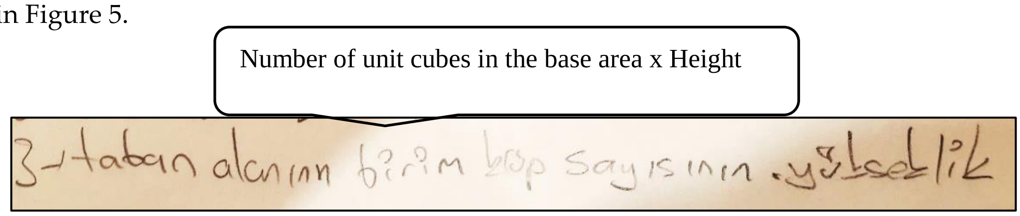


Figure 5. The Volume Measurement Formula Constructed By Ali And Emre on The Worksheet

On the other hand, since Murat constructed the "Surface area of the base x Height" formula, he also constructed another volume measurement formula, "Area of the front surface x Depth", as can be seen in Figure 6, and he shared it in the class discussion.

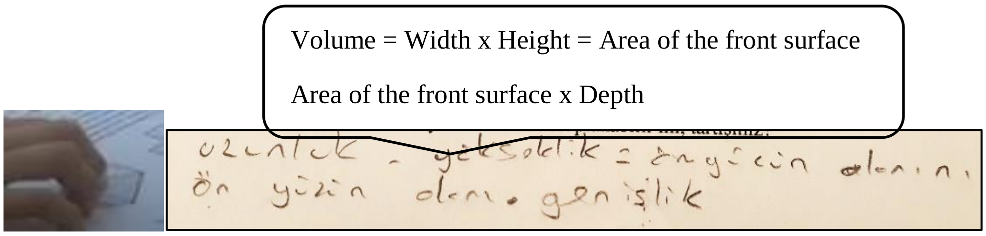


Figure 6. The Volume Measurement Formula Discovered By Murat

As can be seen in Table 9, similar to the case in the class discussions in general, it was once more observed that the students constructed different forms of the "Width x Depth x Height" formula and the "Number of unit cubes in the first layer x Height" volume measurement formula. However, the students emphasized that "Number of unit cubes in the first layer x Height" formula could not be used if the prisms were filled with anything other than unit cubes, such as liquids, and other formulas could be used in any case. Therefore, when the prism was filled with liquid in the fifth week activity, the students' idea that "If the volume of the liquid is not known, the volume cannot be calculated" changed with the discussions held in the classroom. Nevertheless, most of the students were observed to continue to have difficulty about the "Surface area of the base x Height" volume measurement formula. This time, in addition to using technology-supported visual and concrete representations in different activities as can be seen in Figure 7, the teacher noticed that the students associated the term "Base" with the first layer of prisms and, therefore, started to use the expression "Surface area of the base" instead of "Area of the base".

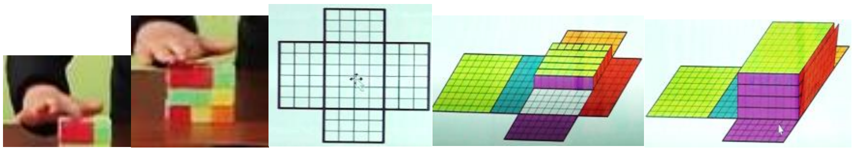


Figure 7. The Teacher's Practices in Class Discussion to Overcome The Difficulty

After that, thanks to the norms *explaining and justifying ideas and mathematical solutions, disagreeing or agreeing with each other* and *inquiring each other, which* emerged as a result of the explanations made and discussions initiated by the teacher during the activities, the students overcame this difficulty in general. The students also reflected the formulas they constructed to their journals. In addition, using visual and concrete representations, the students showed that these discovered formulas applied to all rectangular prisms including cubes and square prisms.

In the ninth week activity, the students were given six daily life problems in which they could use the volume measurement formulas constructed. As seen in Table 9, during the small group and class discussions in this activity, the focus group students and other students in general constructed the "Width x Depth x Height, "Number of unit cubes in the first layer x Height" and "Surface area of the base x Height" volume measurement formulas and they were able to use these formulas within the context of daily life problems. Therefore, the students showed strong indications of overcoming the difficulties they experienced in their shift from three to two dimensions. In this process, during the small group discussion, Murat inquired Ali and Emre asking, "Where is the base area?" as a question about the surface area of the base, and he asked them to show the surface of the base. Ali and Emre, often showed the surface in the questions related to the surface area of the base and associated the surface area with unit squares rather than unit cubes. It was observed that the focus students often exhibited the norms *explaining and justifying ideas and mathematical solutions, disagreeing or agreeing with each other* and *inquiring each other* during the group and class discussions. During the class discussion in this activity, for the problem of how many times the volume of a cube would increase by when one edge of it was doubled, Murat first took a unit of cubes and then thought of creating a cube with a two-unit edge by placing unit cubes next to and on each other so that each edge would be doubled. Then he stated that there was a need for eight two-layer unit cubes and four unit cubes in each layer and, therefore, the volume would increase by eight times. On the other hand, in the small group and class discussions, regarding the problem with a volume of 192 unit cubes and a height of 6 units, Murat provided *an acceptable mathematical explanation and effective solution* by saying, "If it has 6 units, it has 6 layers. In order to find the number of unit cubes on a layer, we divide 192 by the number of layers; that is 6 and then we have 32 units in the first layer. The question asks us the surface area of the base… the base surface of the first layer gives us surface area of the base. We look at the surface for the base area and measure the surface with unit squares. So the answer is 32 units square" and then he wrote this solution on the worksheet as seen in Figure 8. The *acceptable mathematical explanations and effective solutions* made by Murat and other students contributed to students' more meaningful construction of the "Surface area of the base x Height" volume measurement formula.

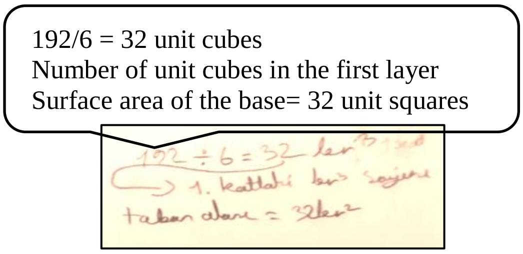


Figure 8. Murat's Problem Solving Process

Final Clinical Interviews

The findings related to the physical-mental actions exhibited by the students during the final clinical interviews conducted after the five-week third stage instructional sequence, are presented in Table 10 under the main theme of "measuring the volume of rectangular prisms".

Table 10. The Physical-Mental Actions Exhibited by the Focus Students During the Final Clinical Interviews

Measuring the Volume of Rectangular Prisms
✓ *Volume measurement formulas*
-Measuring by using layering and ordering strategies (A, E, M)
-Width x Depth x Height (A, E, M)
-Number of unit cubes in the first layer x Height (A, E, M)
-Surface area of the base x Height (A, E, M)
-Measuring by using the Area of the front surface x Depth formulas (A, M)

As seen in Table 10, during the final clinical interviews after the third five-week instructional sequence, while Ali and Murat reflected all the volume measurement formulas discovered in the small group and class discussions, Emre reflected all the formulas except for the "Area of the front surface x Depth" formula, which Murat discovered. The focus group students were able to justify how they constructed the volume measurement formulas in a similar way. They explained the volume measurement formulas they constructed by using the layering and ordering strategies on visual and concrete representations they constructed. During the final clinical interviews, in order to explain his justifications for the formulas, Ali constructed the dimensions of the rectangular prism as seen in Figure 9.

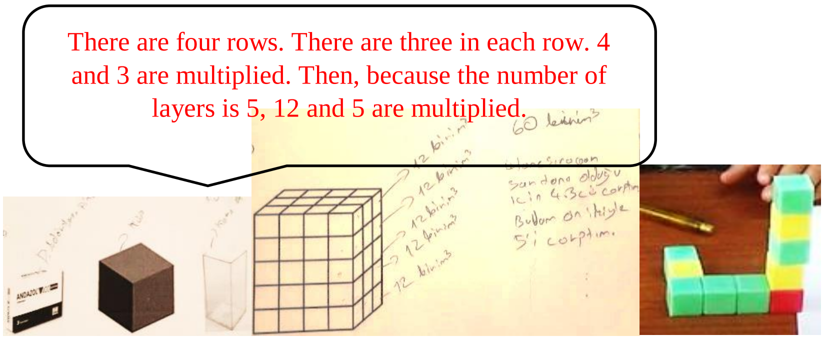


Figure 9. Ali's Physical/Mental Actions About the Volume Measurement of The Rectangular Prism

For this subject, the explanations made by Ali, the focus group student with low achievement, about his physical/mental actions are presented below as an example.

> **Researcher:** How do you calculate the volume of these prisms, which you have called as a rectangular prism, a cube and a square prism?
>
> **Ali**: I multiply the depth by the width and then I multiply the height by the result I found.
>
> **Researcher**: Why is that? How do you explain this formula?
>
> **Ali**: If we fill these prisms with unit cubes, as in this photo, there are three unit cubes in a row. There are four rows and I multiply the width by the depth because there are three in each row and the result is 12. Because I have the same number on each layer, I multiply the result I found by the height, I mean five units and the result is 60 unit cubes.
>
> **Researcher:** Why did you multiply it by the height?
>
> **Ali**: I did because it was the number of layers (he calculated the volume by using the layer and ordering strategies in a structure built with unit cubes, as seen in the image above, and he explained the formula by constructing the dimensions of the structure with unit cubes).
>
> **Researcher:** How else can it be calculated?
>
> **Ali**: I find the surface area of the base by multiplying the depth by the width. I multiply the height by surface area of the base.
>
> **Researcher**: How could it be? Can you explain?
>
> **Ali**: Here, for example, we multiply the width by the depth and we get the number of unit cubes in the first layer. Like what I did here, for example, 12 unit cubes, the bottom surface of the first layer, make 12 unit squares and then because the height is the number of layers, we multiply it by the height. Again, the result is 60 unit cubes (he showed the base of the rectangular prism in the visual image on the concrete model and the rectangular prism visual representation constructed with unit cubes, and he explained the formula).
>
> **Researcher:** Are there any other ways?
>
> **Ali:** Yes, there are. If these prisms are filled with unit cubes, we multiply the number of cubes in the first layer by the height and find the volume. For example, in this picture again, there are 12 in the first layer. If we multiply it by the height, 5, the number of layers, the volume will be 60 unit cubes because there will be the same number of unit cubes in each layer.

Researcher: What if it was not filled with unit cubes?

Ali: Then this formula would not apply.

Researcher: Do you know any other alternative?

Ali: We can also multiply the width by the height and find area of the front surface, and then we can multiply it by the depth and get the same result. For example, here the width is four units and the height is five units. If we multiply them, the front surface area will be 20 unit squares. Then if we multiply it by the depth, three units, it will be 60 unit cubes (he showed the width, height and the front surface on the rectangular prism model, explained the formula and showed all the volume measurement formulas on the worksheet).

As a result, at the end of the process, the three focus students were able to construct the volume measurement formulas for rectangular prisms and to justify how they constructed the formulas with similar explanations.

Conclusion and Discussions

Conclusion and discussions about the teaching experiment process

Prior knowledge of any subject is an important factor in the acquisition and construction of new knowledge in the learning process. Prior knowledge of students is also taken into account while determining the purpose of learning in HLT, an instructional tool (Simon, 1995). The initial clinical interviews conducted to determine the prior knowledge of the three focus students showed that they had both similar and different difficulties in terms of distinguishing between and correlating two- and three- dimensional objects, identifying the bases and dimensions of prisms, counting and building in structures with unit cubes. Ali, in particular, had these difficulties, which suggested that he had weak spatial skills. Indeed, This suggestion is supported by the fact that spatial skills are defined as mental skills associated with the ability to visually understand, use, reconstruct, and interpret relationships (Tartre, 1990). Concrete models are considered to be important tools in the development of spatial and geometric thinking (Clements & McMillen, 1996). Therefore, this might have been caused by his low interaction with concrete models in classroom practices or being away from objects used in daily life. This situation was taken into consideration in the preparation of the teaching activities, and concrete, visual and daily life representations were frequently used in the classroom practices. On the other hand, students' perceiving unit cubes as "small" reveals that concrete objects should be used carefully. The students' construction of unit cubes as "small" might have resulted from the use of concrete object models representing unit cubes in classroom environments. The cause of the difficulties in identifying the bases and dimensions of prisms might be the use of interchangeable or ambiguous concepts in Turkish language that can lead to misunderstanding. For example, the students probably associated some words in everyday Turkish with words related to prisms such as the base of prisms with "Ceiling and floor" (e.g., the ceiling and floor of a room) and the height and width of prisms with the "Height of a person". In fact, even the results from studies conducted with teacher candidates or teachers are in parallel with the results obtained in this study. For example, Gökkurt and Soylu (2016) found that secondary school mathematics teachers had difficulty in defining the prism concept and identifying its basic elements. Also, Bozkurt and Koç (2012) emphasized that teacher candidates could not use the language of mathematics adequately in defining prisms and, therefore, they had difficulty in defining prisms.

Although not as much as Ali, Emre and Murat also had difficulties about counting and building structures with unit cubes and made some mistakes, which is in parallel with various research findings (Hirstein, 1981; Ben-Chaim et al., 1985; Olkun, 2003). The students used different strategies depending on whether the structures were simple or complex, as stated in some research findings in the literature

(Battista & Clements, 1996; Olkun, 1999) rather than using consistent strategies that would lead them to the correct conclusions and would not vary from structure to structure. This result could have been caused by the students' insufficient experience with structures built with unit cubes. In addition, as Olkun (1999) suggested, difficulties in perceiving three-dimensionality and structural regularity in complex structures might have led to this situation.

One of the main results obtained from this study is that all the three focus students overcame the difficulties they experienced in the initial clinical interviews in the first and second stage instructional sequences. While building a rectangular prism with unit cubes *by discussing with each other and inquiring each other* during the small group discussion in the fourth week activity in the second stage instructional sequence, the students first constructed the dimensions and completed the remaining parts, which supports this situation. Also, while calculating the number of unit cubes in the rectangular prism they built, the students discovered and used the "Width x Depth x Height" *mathematical solution strategy*. This result can be considered as a remarkable finding indicating that the students' perception of three dimensions developed and they were able to think spatially about these structures. However, during the small group and class discussions in the fifth week activity, the focus group students said, "When a rectangular prism is filled with liquid, the volume of the prism cannot be calculated when the volume of the liquid is not known", which implies that the "Width x Depth x Height" strategy was not constructed by the students as the volume measurement formula. Battista and Clements (1996) stated that it is difficult for students to make sense of the volume structure by means of the formulas given directly and that this means nothing other than memorization. Similarly, Zembat (2009) argued that use of the "Depth x Width x Height" formula based on memorization without comprehending the principles behind the formula causes students to move away from the mathematical structure. For this reason, Battista (2007) emphasized the importance of understanding and internalizing mathematical concepts while teaching the concept of volume. Olkun (2003) recommended that mathematics teaching should be carried out with activities that enable students to find the formula and rules related to volume measurement and to construct the basic concepts on their own. Also, Zembat (2007) suggested that teaching lessons should be designed in such a way that students can make reflective abstraction. One of the results of this study is that the third stage instructional sequence activities designed based on these research findings and suggestions in the literature play a key role in mathematics teaching and learning, as Simon and Tzur (2004) emphasized. We could therefore suggest that mathematical activities provide valuable tools in the teaching process carried out for the hypothesis that students can create volume measurement formulas and perform reflective abstraction. In this regard, the easiest way to determine the measurement result in volume measurements is demonstrated by the Cavalieri's principle. The essence of the Cavalieri's principle lies in the fact that the volume of objects can be determined by dividing those objects into layers (Zembat, 2009). As anticipated in the seventh week activity, which was designed with this perspective in mind, it was Emre that first constructed the "Width x Depth x Height" volume measurement formula during the small group discussion in this activity. While constructing this formula, Emre counted the unit cubes by using the bottom-to-top strategy as also thought by the other students in the class in general. Murat, on the other hand, discovered the "Height x Depth x Width" volume measurement formula by using the strategy of counting unit cubes by separating them from the side. Murat's discovery of the formula in this way is interesting considering the fact that most of the students tended to count the unit cubes by using the bottom-to-top strategy while Murat preferred counting the unit cubes by starting from one side and expressed the formula in a different way. On the concrete representation of the rectangular prism that the group built using unit cubes, the mental actions that Murat built with the structure as a result of his clear view of the unit cubes on the side could contributed to his discovery of this formula. Immediately after the discovery of these formulas, Emre discovered the "Height x Width x Depth" formula, too. While discovering this formula, Emre counted the unit cubes using the front-to-back strategy this time. We could suggest the formula constructed by Murat activated Emre's *mental actions* and paved the way for him to discover this formula. Hence, prior to their discovery of these formulas, we could suggest that the focus students' experiences as a group in calculating the volume of a rectangular prism made of unit cubes on the visual representation using layering and ordering strategies and constructing its concrete representation with unit cubes and the *discussions they had with each other* on it helped the students discover and construct these formulas. Meanwhile, the students in another group constructed a volume measurement formula expressed as "Width x Depth = Area of the first layer and Area of the first layer x Height". Another interesting result in this study was that the students generally associated surface area of the base with the number of unit cubes on the first layer rather than the unit squares in the base surface. However, in the eighth week small group discussion, Murat, who constructed the Surface area of the base x Height" formula correctly in his mind, used the *mental knowledge* he obtained from this formula and discovered the "Area of the front surface x Depth" formula, a strikingly different volume measurement formula which was not even mentioned in the textbooks. In addition, during the ninth week activity, for the problem of how many times the volume of a cube would increase by when one edge of it was doubled, Murat again came up with a remarkably *different mathematical solution strategy* that he constructed by putting the unit cubes side by side and on top of each other. It can be said that this strategy of Murat shows that his spatial skills were developed and that he deeply abstracted the measurement of volume. Furthermore, Murat's *acceptable mathematical explanations and solutions* in the small group and class discussions for calculating surface area of the base in the problem with a volume of 192 unit cubes and height of 6 units were important. This was another indication that Murat constructed the volume of rectangular prisms. In addition, such an effective solution contributes to constructing the "Surface area of the base x Height" volume measurement formula.

As another important result from this study, in the final clinical interviews conducted at the end of the third stage instructional sequence, the focus group students were able to explain and justify how they constructed the volume measurement formulas as recommended in the relevant literature. According to the Cavalieri's principle, what repeats in "Width x Depth x Height and Surface area of the base x Height" volume measurement formulas is the volume of the base layer, not surface area of the base (Zembat, 2009). As a matter of fact, all the three students showed that they understood the idea underlying these formulas by making a distinction between the surface area of the base and the number of unit cubes in each layer. In addition, it was noteworthy that during the final clinical interviews, Ali constructed the dimensions of the rectangular prism in order to justify his actions regarding the volume measurement formulas. This remarkable act of Ali suggests that Ali's spatial skills regarding rectangular prisms had improved by the end of this process.

As a result, this study provided important results regarding recognition of rectangular prisms, basic properties of prisms, structures built with unit cubes and volume of rectangular prisms. This situation indicates the originality of the research, and these findings could both provide useful practical information to the teachers teaching in the lessons and contribute significantly to future research on these subjects.

Conclusion and Discussions about the Focus Students' Mathematical Abstractions

The learning principles of Piaget, which were emphasized by Gallagher and Reid (1981), were taken into account in exposing the abstraction mechanisms of students regarding volume measurement in rectangular prisms. In this regard, based on the principle that Piaget first accepted the competence as a prerequisite for learning, the initial clinical interviews were conducted and the students' strengths and weaknesses about these points were identified. Due to the weaknesses detected in the students, lesson plans were prepared for this within the framework of HLT and efforts were made to ensure the competence of students in these subjects.

Piaget argued that learning is an internal construction process. Based on this thought, the activities related to volume measurement formulas and the actions in these activities were designed by considering the progress of the internal process. According to Piaget, the active knowledge that the individual creates as a result of his own internal construction consists of cognitive structures created by the functioning of the individual's learning mechanism. Therefore, the active knowledge consisting of cognitive structures of the focus students as a result of class teaching activities were explored during the final clinical interviews.

Piaget emphasized that learning takes place through conflicts, inquiries and new arrangements made in the mind as a result of these situations. The new arrangements made in the mind generally activate social interaction. In other words, Piaget stated that social interaction is an important factor in learning, but focused more on the cognitive dimension of learning. This study, on the other hand, focused on how the students mathematically abstracted information, as well as social factors that supported abstraction in the process. As a matter of fact, it was emphasized in many studies in the literature (Bauersfeld, 1980; Cobb & Yackel, 1996; Cobb, et al., 1997; Yackel et al., 1991, 1993; Yackel & Cobb, 1996) that class social interactions support learning. For this reason, during the instructional sequences in the study, an environment was created in which students could interact with each other and with the teacher. As revealed in the results section of this study, it was observed that the students adopted and frequently employed various social and sociomathematical norms in this process, such as *trying to listen to and understand each other, inquiring each other, explaining and justifying their ideas and solutions, agreeing or disagreeing, making acceptable mathematical explanations and justifications,* and *making mathematical solutions*. The focus group students Ali, Emre and Murat frequently asked each other to *explain their ideas, make mathematical solutions* and *justify their solutions* in the small group discussions. Especially during this process, Emre and Murat asked Ali, "What do you think?", "What is your solution? Can you explain?", "Did you understand our solutions?" and "Do you agree with our solutions?". These examples of peer inquiring supported the development of Ali. As Piaget emphasized, this kind of interaction with Ali's peers was a critical factor in his cognitive and social development. Indeed, Smith et al. (2009) found that peers' discussions with each other increased comprehension and positively supported student performance. Similarly, teacher-student interactions were ensured in the class discussions, and the teacher played a guiding role and asked all students *to explain their thoughts, to make mathematical solutions, to justify the solutions they made, to express points they did not understand, to inquire each other,* and *to disagree with each other when they did not agree with their friends.* During the teaching process, the focus group students, especially Ali, made a significant progress by participating in activities in the teaching process at an increasing level. In addition to their own individual mental actions, these norms exhibited in classroom practices played a supportive role in the mathematical abstractions of the focus group students. As a matter of fact, in parallel to our results, many studies (Cobb, 1989, 1990; Cobb, et al., 1991; Cobb & Yackel, 1996; Wood et al., 1995; Yackel & Cobb, 1996) emphasized that social interactions are important in learning along with actions.

Piaget emphasized that regulating his or her actions at a higher level is an important factor that facilitates learning for an individual. In this sense, the focus students organized their activities at a higher level by reaching the "Surface area of the base x Height" formula based on the conclusion that the "Width x Depth" formula was equal to the expression "Surface area of the base" after discovering the "Width x Depth x Height" formula. Similarly, Ali and Murat's discovery that "Width x Height" formula was equal to the expression "Area of the front surface" can be considered as another example that they organized their actions at a higher level. As a result of the final clinical interviews conducted as the last part of the teaching experiment process, the abstraction mechanisms of the focus students for measuring volume in rectangular prisms were revealed. The abstraction mechanisms determined are presented in Figure 10.

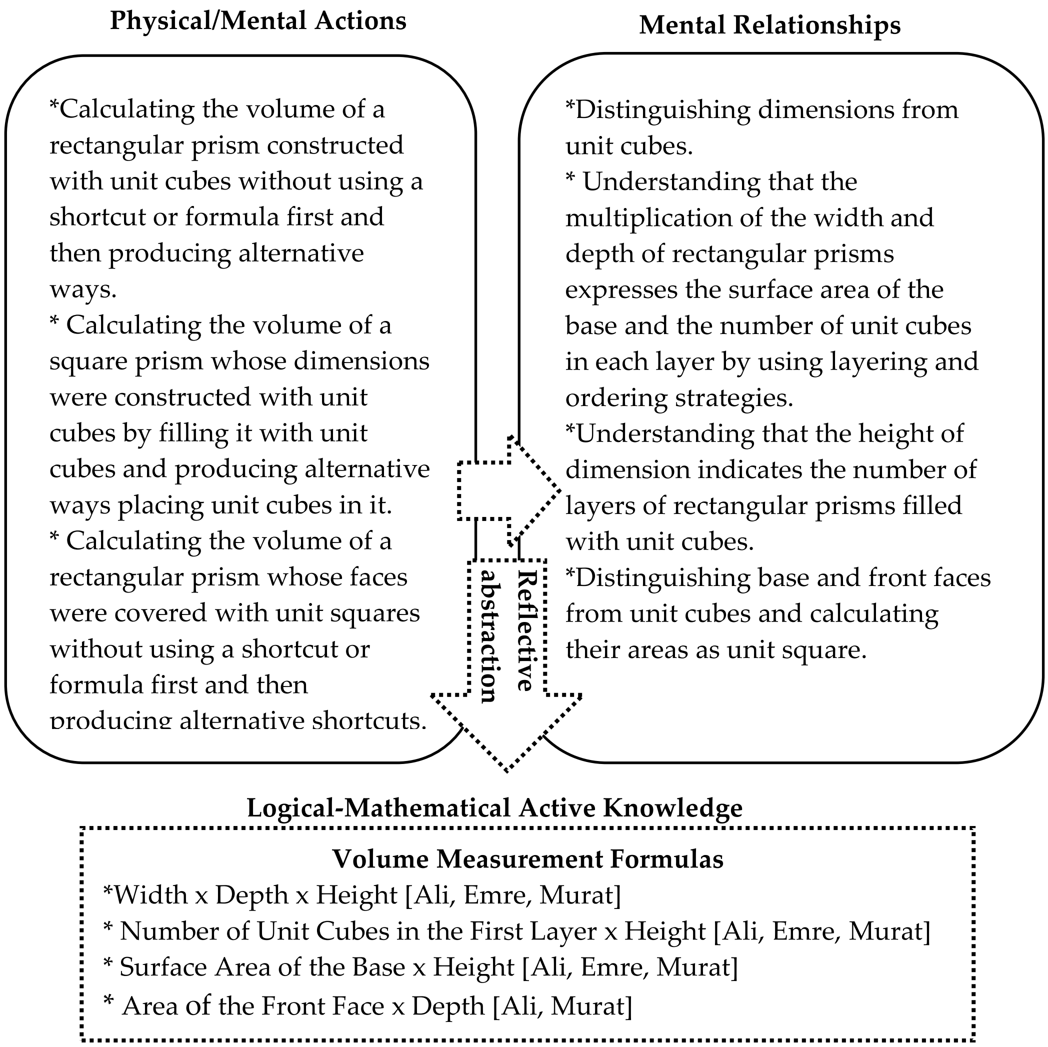


Figure 10. Abstraction Mechanisms of The Focus Group Students

As can be seen in Figure 10, all the three focus students abstracted the volume measurement formulas they discovered and structured in the teaching process of rectangular prisms and the underlying principles of formulas by establishing mental relations at a reflective (based on thinking) level. Using the "Width x Depth x Height" and "Number of unit cubes in the first layer x Height" formulas, the students managed to construct the mental relationships suggesting that multiplying the number unit cubes in each row of each layer by the number of rows would be equal to the number of unit cubes in each layer and the height would be equal to the number layers. They also discovered the difference of the dimensions of rectangular prisms from unit cubes. In addition, using the "Surface area of the base x Height" volume measurement formula, they were able to establish mental relationships suggesting that multiplying the depth and width of prisms would wield the number of unit cubes in each layer and surface area of the base. Similarly, Ali and Murat obtained the "Area of the front surface x Depth" volume measurement formula by suing the mental relationship suggesting that multiplying the width and height of rectangular prisms would also yield the area of the front surface. They also managed to switch from three dimensions to two dimensions by constructing surface areas as unit squares by distinguishing surface areas from unit cubes.

In conclusion, Zembat (2016) stated that the abstraction mechanism is an important tool in elaborately revealing the process in which mathematical knowledge is created. Dubinsky (1991) and Simon (1995) emphasized that reflective abstraction is the basis for cognitive development and can be a powerful tool for advanced mathematical thinking. In this regard, we could suggest that the knowledge, skills and social experiences acquired by all the three students within the scope of this study can contribute to their mathematical thinking in their future learning lives.

Recommendations

Several recommendations can be made in line with the findings and results obtained from this study. First of all, it was observed that the teaching process carried out within the framework of HLT brought a useful and practical application to the constructivist approach, which is a learning theory, with respect to teaching. Therefore, HLT can be included as a teaching tool in mathematics curriculum. This study also showed that sociological factors besides cognitive factors were supportive in mathematical abstractions of the focus students. Therefore, mathematics learning can be addressed in mathematics curricula within the framework of EP theoretical approach. The learning environment designed in two stages as small group and class discussions in the light of this theoretical framework could offer supportive tools for students with different achievement levels to learn from their peers interactively and especially for those with low achievement level to make progress. Therefore, it is recommended that teachers design group tasks as well as individual tasks in class activities. In this study, while designing the plan of each week, the activities were designed using different materials and representations. It was observed that activities designed using various tools such as visual and concrete rectangular prism representations, unit cube sets used as teaching materials, geomag magnets and sticks, use of technology provided useful tools for the students to learn and facilitated their learning. For this reason, when designing activities, teachers should support these activities with the use of different materials, making it easier for students with different levels of achievement and learning pace to learn. On the other hand, although a teaching experiment designed within the framework of HLT was carried out in this study, an HLT related to volume measurement was not developed in the study. Therefore, another study in which an HLT developed on volume measurement in rectangular prisms can be carried out and contribute to the field.

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