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# A Proposed Conceptual Framework For Enhancing The Use Of Making Connections Skill In Mathematics Teaching *

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## Abstract

The purpose of the present study was to develop a conceptual framework about how to make connections in mathematics courses. To this end first of all, the researchers emphasized the importance of making connections in terms of mathematics and mathematics teaching. Then they delineated how making connection is generally handled in mathematics education. Last, in order to contribute to the operationalization of the connection skill, it was proposed that this skill is composed of four major components; i.) connection between concepts, ii.) connection between/among different representations, iii.) connection with real life, and iv.) connection with different disciplines. Each component was explained by giving examples, and the relationship between these components was examined as well. This study concludes by discussing the benefits of the conceptual framework for mathematics learning and teaching, and proposing conclusions and implications for further research.

## Keywords

Making connections in mathematics teaching

Connection skill

Connection with different disciplines

Connection between different representations

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## Introduction

The core characteristic of the discipline of mathematics is to identify the relationship between mathematical objects, make generalizations depending on this relationship and attempt to prove these generalizations (Yıldırım, 1996). This perspective has caused mathematics to be regarded as a sequential and cumulative discipline in general. What is meant by being sequential and cumulative is that mathematical concepts and systems are constructed on one another, they are interrelated and prior knowledge or concepts are used in defining concepts and constructing systems. For instance, the construction of triangle requires the connection between some mathematical concepts such as point, line, line segment and angle. Furthermore, triangle is also a part of the system of polygons. Therefore, being sequential and cumulative refers to making connections between the concepts in terms of mathematics teaching.

The history of mathematics indicates that making connections has a significant place in the development of the discipline of mathematics. Analytic geometry is the monumental aspect of making

* This study is stemmed from the second author's Master thesis supervised by the first author. This paper presents an expanded version of the proposed framework, which was first employed to analyse the thesis' data. An earlier version of this paper is presented in Turkish Computer and Mathematics Education Symposium (TURCOMAT 2).

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connections in the history of mathematics. As Borceux (2014) puts it, geometric problems were solved using methods which were inherited from Greek mathematicians and generally required the use of rulers-compasses until the 1600s. It was rather difficult to solve geometric problems using these tools and methods. Fermat and Descartes, independently, introduced analytic geometry into the discipline of mathematics in the 1630s (Borceux, 2014), and this paved the way for representation of geometric problems by algebraic symbols and methods, thereby facilitated the solution of the problems more effectively. Analytic geometry originated from the connection of these two fields of mathematics. This subsequently helped the emergence and development of Calculus which affected the history of humankind profoundly with the significant contributions of Newton and Leibniz in the 1670s.

Making connections which is inherent in the nature of mathematics and has a key role in development of discipline of mathematics is the *sine qua non* of mathematics learning and teaching and one of the essential skills that students are supposed to acquire in mathematics teaching (NCTM, 2000). Although this skill is mainly treated as a skill which needs to be acquired by students in mathematics education literature, it is also important whether the teacher makes connections in the classroom and how these connections are made. Tchoshanov (2011) argues that teachers' knowledge of the concepts and the relationships between them positively affect pupils' achievement and the quality of the courses. Furthermore, Tchoshanov (2011) posits that this knowledge is a distinctive element of being a successful teacher and that the teacher managing the process of attaining the hoped-for results is expected to undertake important tasks in this regard. However, a review of the related literature indicates that a comprehensive conceptual framework regarding what making connection is and how this skill can be acquired by students is lacking. In this study, it is therefore aimed to propose a conceptual framework to help teachers about how to make students to obtain this skill in mathematics teaching by determining the components of making connections. This study hence not only attempts to determine the content of making connection skill but also presentsa guiding tool for teachers for making connections in their teaching.

## Making Connections Skill in Mathematics Education

The concept of making connection is generally used in the research conducted under two major themes in mathematics education literature. The first one is the *understanding* theme, and the other one is the *skill* theme. Even though it is not clearly explained, the concept of making connection is investigated in terms of relational thinking and relational understanding (Carpenter, Franke, & Levi, 2003; Carpenter, Levi, Franke, & Zeringue, 2005; Empson, Levi, & Carpenter, 2010; Hunter, 2007; Presmeg, 2006; Skemp, 1976). Relational understanding came to the fore specifically with Skemp's (1976) research in mathematics education literature. Mathematical understanding is divided into *relational understanding* and *instrumental understanding*. Skemp (1976) defined relational understanding as a kind of understanding which covers knowledge of a mathematical operation and its reason, associated it with a network of conceptual relationships and proposed that this kind of understanding involves the features of one concept and its relationship with other concepts. For example, interpreting the concept of derivative as i.) slope of the tangent drawn in a function at a particular point, ii.) the function's instantaneous rate of change at that point or iii.) the limit of difference quotientat that point can be considered as relational understanding of the concept of derivative. In addition, knowing the relationships between the concept of derivative and the concepts of anti-derivative (i.e. integral), limit and function can be given as an example of relational understanding. Similarly, knowing that a square is a special kind of rectangles shows the existence of relational understanding. Skemp (1976) argued that such relational understanding can lead to meaningful and enhanced learning. He also underlined that relational understanding should be emphasized alongside with instrumental understanding in teaching.

Skemp (1976) defined instrumental understanding as the ability to carry out mathematical operations using rules without knowing the underlying reasons and applying these rules. To give an example, when a student finds the answer of $1{:}\frac{1}{2}$ division operation by inversing the latter fraction and multiplying it with the former one ($1x\frac{2}{1}$), it does not suggest that the student knows the meaning of division in fractions exactly. Here relational understanding requires knowing how many halves (1/2) of which the whole consists, and that makes the answer 2. Similarly, finding the derivative of $f(x) = x^2$function as $f'(x) = 2x$ without knowing the meaning of the relationship between these two can be given as an example to instrumental understanding.

This kind of understanding specified as relational understanding by Skemp (1976) requires relational thinking as well. Relational thinking is closely linked with relational understanding, and it is the way of mind's working in mental dimension. At the same time, relational thinking and understanding are associated with conceptual understanding (Hiebert & Lefevre, 1986) and meaningful learning (Ausubel, 1968) which are constantly emphasized in the process of mathematics learning and teaching.

Making connections has become one of focal points of mathematics teaching curricula as a skill or a standard together with the concepts of relational thinking and understanding. Accepting that children can learn deeply when making connections both within mathematics and with other fields (NCTM, 2000) has brought connection skill into the forefront in mathematics learning and teaching. At this point, the main question to be answered is: "What does making connections skill mean?" Related literature indicates that some definitions are made specifically in policy documents in accordance with relational understanding. In the process standards of NCTM (2000), for instance, with regard to making connections, students are expected to:

- *Recognise and use connections among mathematical ideas;*
- *Understand how mathematical ideas interconnect and build on one another to produce a coherent whole;*
- *Recognise and apply mathematics in contexts outside of mathematics.*

In parallel with NCTM's (2000) perspective, the following explanation was made for connection skill in the 5-8th grade middle school mathematics teaching curriculum which was revised in 2013:

> *There are connections between mathematics, other disciplines and real life. Therefore, connection skill includes making connections between mathematical concepts and one mathematical concept's relationship with other disciplines and daily life. Moreover, it is important that mathematical operations are connected with the underlying concepts* (MoNE, 2013, p. 5).

Given the review above, it is seen that some common components (such as making connections with real life) about making connections skill are highlighted. However, connection skill needs to be analyzed systematically in order to clearly understand what making connections skill means. Drawing from the relevant literature (NCTM, 2000; MoNE, 2013), the researchers proposed making connections skill as a skill comprising four major components. These components make up the content of making connections skill and demonstrate how this skill is conceptualized and defined in this study. The four major components are presented in the following table.

**Table 1.** A Conceptual Framework For Making Connections Skill

| |
|---|
| **a.) Connections between concepts** |
| $a_1$. Making connections between the concept and others |
| $a_2$. Making connections between the concept and its sub-concepts and among sub-concepts |
| **b.) Connections between different representations of the concept** |
| **c.) Connections with real life** |
| $c_1$. Teaching the concept within real life context |
| $c_2$. Stating real life connection via verbal examples |
| **d.) Connections with different disciplines** |
| $d_1$. Teaching the concept within a different discipline's context |
| $d_2$. Stating the connections with other disciplines by giving verbal examples |

In what follows, each component is defined, and the content of making connections skill is determined. Before explaining what each component means, it seems useful to delineate what the terms used for the concepts refer to. The concepts such as derivative, functions and ratios which have definitions are directly used as concepts. As mathematics teaching is highly dynamic and complex, alongside main concepts, it is required to use many mathematical ideas, facts and procedures. Hiebert et al. (2003) accept the potential relations among mathematical ideas, facts and procedures as a part of making connections skill. Within the context of this study, mathematical facts and procedures such as the straight line $y = 2x + 1$ and the function $f(x) = x^2$ are considered asconcepts as connections can be made among them as well. This preference is taken up especially for the sake of effective communication throughout the paper.

***1. Connections Between Concepts***

Mathematics is a field of continuous inquiry about new relationships and of proving these relationships. In this process, connections between concepts and making connections are indispensable. This structure emerging in the construction of mathematics is valid for mathematics teaching and learning. Learning a new concept necessitates making connections between already learned concepts or the concepts which have turned into prior knowledge. Thus, prior knowledge of the concepts must be analyzed in detail before moving to another topic (Hasemann & Mansfield, 1995). Some of this prior knowledge is the concepts to which the concept is related. This issue is particularly taken into consideration while preparing the curricula and deciding the places of the outcomes.

The importance of making connections between the concepts in the process of teaching a mathematical concept is vastly highlighted in related literature (MoNE, 2013; NCTM, 2000; Van de Walle, Karp, & Bay-Williams, 2012). Connections between concepts are usually seen in the research conducted under the titles of conceptual understanding, relational understanding and concept maps (Baki & Şahin, 2004; Carpenter et al., 2005; Empson et al., 2010; Johnson, Seigler, & Alibali, 2001; Skemp, 1976; Watson, 2004). As concept maps require making connections between concepts (Hasemann & Mansfield, 1995; Williams, 1998), and conceptual and relational understanding requires constructing a network of relationships between concepts, it is not unusual that connections between concepts are mostly seen in the studies carried out under these themes.

In this study, connection between concepts is investigated as a component of making connections skill, and it is accepted to have two sub-components. Although there is not a distinction in the related literature, such a distinction is used to fully grasp the connection between concepts:

- Making connections between the concept and others
- Making connections between the concept and its sub-concepts and among the sub-concepts

***1.1. Making Connections Between The Concept And Others***

Making connections between the concept and others refers to connecting one mathematical concept or statement with different mathematical concepts. While teaching identities, making connections with the concept of area can be an example of this component. In teaching the identity $(a + b)^2 = a^2 + 2ab + b^2$ calculating the area of a square whose edge length is $a + b$ unit and finding that the area equals to the identity exemplify how connections can be made between different concepts. As can be seen in this example, one can benefit from the concept of area which is a geometric concept in teaching an algebraic concept. A similar example can be given for the connection between derivative and integral. In fact, finding the integral means finding the primitive of a function whose derivative is already found, and therefore, integral is anti-derivative. These are also examples of the connections between two concepts and making such connections. Moreover, dividing a rectangle through its diagonal into two parts and finding the formula of right-angled triangle as a result is another example of making connection between two concepts.

The discipline of mathematics is a sequential and cumulative field, and it is constructed upon the relationships between concepts due to its nature. Thus, making connections between concepts can be regarded as usual. However, as is shown in the example about the identity above, making connections between the concept of area and identity which is an algebraic statement may lead to better learning. For this reason, the researchers hoped to draw attention to the necessity of making connections, which is already embedded into the nature of mathematics, in practice by examining making connections in the context of concepts' relationship with other concepts.

***1.2. Making Connections Between The Concept And Its Sub-Concepts And Among The Sub-Concepts***

The relationship between a concept having a more general characteristic and its sub-concepts and the interrelationships between the sub-concepts of one concept are examined within the context of this component. The concept of triangle can be illustrative of this sub-component. Triangles are classified into groups according to their sides and angles. Based on their angles, triangles are classified as acute, obtuse and right triangles. They are classified by their sides as isosceles, equilateral and scalene triangles. With this classification, new sub-concepts regarding triangles emerge. Therefore, there is a relationship between the concept of triangle which is a more general concept and the obtuse triangle which is a sub-concept of the triangle. On the other hand, relating the acute triangle with the equilateral triangle and revealing that every equilateral triangle is an acute triangle can be an example of the connections between the sub-concepts of one particular concept. Likewise, an isosceles triangle can be acute, obtuse and right angled, which is assessed in terms of this component as well.

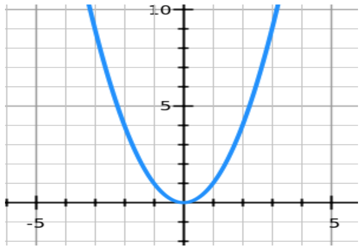
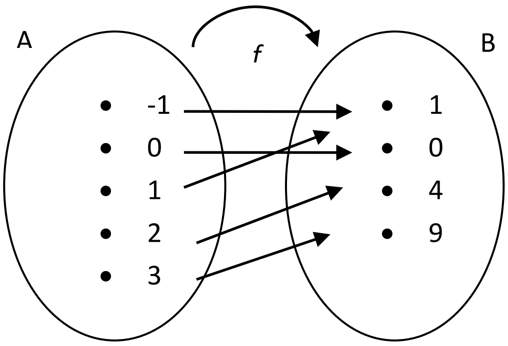
***2. Making Connections Between Different Representations Of The Concept***

Knowledge of mathematical representation is accepted as one of the process standards by NCTM (2000). As it is closely related to connection skill, making connections between different representations of one concept is considered as a component of connection skill in this study. The connection between different representations and connection skill is significant because demonstrating one concept using different representations helps the concept to be understood better and enhances the network regarding the concept in the mind. At the end of this process, relational understanding cones out (Van de Walle et al., 2012, p. 27).

Different representations and transformation between them are broadly examined in the related literature (Ainsworth, 1999; Lesh & Doerr, 2003; Ainsworth & Van Labeke 2004; Bingölbali, 2010; Duncan, 2010; Van de Walle et al., 2012). Some mathematical representations, in particular, come to the forefront: i.) verbal statement, ii.) concrete manipulatives (number counters, fraction bars, real models, etc.), iii.) pictures or diagrams (number lines, area models, etc.), iv.) written symbols, v.) tables, vi.) graphics, vii.) equation and viii.) figures. More different representations exist in advanced mathematics. These representations and connections between them are considered within the context of connection skill in this research. Therefore, development of this skill depends on making strong connections amongst different representations.

Mathematical representations may differ according to the concept under investigation. To illustrate, although there are concrete representations for fractions, it is not possible to use a concrete representation for the concept of group which is a more abstract concept. Therefore, characteristics and limitations of concepts are determinants of the representations to be used. The example presented in Table 2 demonstrates different representations for the statement $f(x) = x^2$. Different representations are exemplified, and specific representations for the concept emerging due to the nature of the statement are examined based on this example. We can name the representation "$f(x) = x^2$" for the function of$f(x) = x^2$ as algebraic or analytical or written symbol representation. The table below indicates how differently this statement can be shown via multiple representations.

**Table 2.** Different Types Of Representations For The Function $f(x) = x^2$

<table>
<tr><th>Type of Representation</th><th>Example</th></tr>
<tr><td>1.Algebraic representation</td><td>$f(x) = x^2$</td></tr>
<tr><td>2.Table/Numerical representation</td><td><table><tr><td>$x$</td><td>...</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>....</td></tr><tr><td>$f(x) = x^2$</td><td>...</td><td>1</td><td>0</td><td>1</td><td>4</td><td>9</td><td>....</td></tr></table></td></tr>
<tr><td>3. Graphical representation</td><td>
</td></tr>
<tr><td>4. Verbal representation</td><td>The rule matching every real number with its square</td></tr>
<tr><td>5. Matching between two sets representation</td><td>
</td></tr>
<tr><td>6. Set builder representation</td><td>$f = \{\dots, (-1,1), (0,0), (1,1), (2,4), (3,9) \dots\}$<br>ya da $f = \{(x, x^2) | x \in R\}$</td></tr>
</table>

Given the table including the statement $f(x) = x^2$, it is seen that the same statement has different types of representations. When these representations are used and the relationships between them are constructed, it is clear that the concept and the statement can be better understood[3]. Emphasizing these types of representations and the relationship between them in teaching can contribute to the development of the students' connection skills. Thus, different representations regarding concepts and statements, as is seen in the example of $(x) = x^2$ , are necessary for conceptual and meaningful learning, and these are accepted as a main component of the connection skill in the current study.

### *3. Making Connections With Real Life*

The relationship between mathematics and real life is frequently emphasized in the process of learning mathematics in the related literature (NCTM, 2000; Jurdak, 2006; Stillman & Galbraith, 1998). This kind of connection is examined in terms of two sub-components in this study:

- Teaching the concept within real life context
- Stating real life connection via verbal examples

#### *3.1. Teaching The Concept Within Real Life Context*

Teaching the concept within real life context means teaching mathematical concepts using real life situations. Real-life contexts cover not only the situations students face in real life directly but also

[3] To present the representations economically, matching between a limited numbers of reel numbers is provided. However, the aim is that all reel numbers are to be matched with their squares in the same manner.

the problem situations they might imagine (Van den Heuvel-Panhuizen & Wijers, 2005). Due to the fact that the contexts which an individual may encounter with in real life are limited, real-life contexts are considered as the situations faced or likely to be faced. The following examples are illustrative of real-life contexts: Using analogies such as elevator-ground floor- basement floor in teaching negative numbers; and benefitting from students' mathematics written exam results as actual data in teaching the concepts like mean and median; and using the concepts of age and pool mostly faced in classical verbal problems (Gainsburg, 2008). Because typical story problems help the construction of the relationship between real life and mathematics, these are specifically preferred by teachers in mathematics teaching (Ji, 2012).

It is useful to underline that this component is different from the component of making connections with different disciplines. Concepts which belong to a different discipline other than mathematics are used to make connections with different disciplines. However, real-life contexts (i.e. using elevators as an example for negative numbers) are directly chosen and used for teaching mathematical concepts while constructing the connection with real life contexts.

***3.2. Stating Real Life Connection Via Verbal Examples***

Stating the relationships between mathematics and real life refers to using real-life contexts merely verbally while teaching a concept. An example for this sub-component can be that the teacher can give only verbal examples from the classroom and daily life while teaching the concept of rectangle after the concept has been presented mathematically. Similarly, the teacher can present the concept of ratio formally and give a verbal example like; "the ratio of the number of girls to the number of boys can be easily found by this concept", instead of teaching the concept of ratio by using the ratio of number of the girls to the number of boys in the classroom.

Related literature demonstrates that whether teachers make connections with real life verbally or through a given context is not being revealed. However, Coşkun (2013) concluded that teachers tended to only mention the connections with real life in some cases. Furthermore, it was revealed that problems were used as real-life contexts in general in that study. This component is therefore important to show that the connections with real life can be made in different ways, and it is taken into consideration in the conceptual framework proposed for making connections in this study.

***4. Making Connections With Different Disciplines***

The concepts belonging to different disciplines can be benefitted from while teaching a mathematical topic or concept. Using the concepts of different disciplines is a part of the component of making connections between mathematics and different disciplines. Connections with different disciplines can be made using two sub-components:

- Teaching the concept within a different discipline's context
- Stating the connections with other disciplines by giving verbal examples

***4.1. Teaching The Concept Within A Different Discipline's Context***

In this component, a mathematical concept or statement is taught within the context of a different discipline, and the concepts of different disciplines are employed. For instance, the teacher can make use of the concepts of latitude (parallel) and longitude (meridian) in the geographical coordinate system of Turkey to teach the concept of coordinate system. Turkey is located between the 26°-45° eastern meridians and 36°-42°northern parallels. Thus, the concepts of latitude and longitude which are geographical concepts are used to find out the geographical position of Turkey. The coordinate system is needed to find the position of one point on plane in mathematics. The concepts of abscissa and ordinate which are mathematical concepts are used in the coordinate system in order to find the position of one point (e.g. (3,5) point), namely to locate its position. Briefly, the teacher can benefit from the geographical concepts of latitude and meridian to teach the concepts of abscissa and ordinate. This can be an example for making connections with different disciplines. As presented in the example, the concepts of different disciplines can be used to teach mathematical concepts.

The context presented in the example above is different from real-life contexts and includes specific concepts of a different discipline. The coordinate system, for example, can be examined by making connections with the positions in a cinema or classroom. These contexts are not discipline-specific, and they are therefore included in the component of real-life contexts. Speed problems used in mathematics set another example for making connections with different disciplines. To illustrate, the concept of proportion can be used to solve the problem: "if one car travels a distance with a speed of 80km/h in 4 hours, how many hours would it take for this car to travel the same distance with the speed of 100 km/h? " The concept of speed belongs to the field of science and is used to teach or reinforce the concepts of ratio and proportion. Therefore, using one concept included in science can be considered for the component of making connections with different disciplines.

### *4.2. Stating The Connections With Other Disciplines By Giving Verbal Examples*

This sub-component covers using the concepts of different disciplines verbally while teaching mathematical concepts. Teaching negative numbers can illustrate this connection for this sub-component. After negative numbers have been taught mathematically, a connection can be made with thermometer and temperature from science. Likewise, instead of teaching derivative in a context requiring calculation of instantaneous speed, the concept can be taught formally and then an example like; 'instantaneous speed and acceleration can be calculated using this concept' can be given.

It is interesting that there is no distinction in the literature about whether teachers use making connections with different disciplines verbally or in particular contexts. Although making connections with different disciplines is encouraged in the curricula, Coşkun (2013) found that teachers benefit from making connections with different disciplines at a very low level in real classroom contexts. This sub-component is considered to be significant as it demonstrates that connections can be made with different disciplines in different ways, and the researchers therefore considered this component within the proposed conceptual framework.

The four components of making connections are examined separately to this point. The following table demonstrates the four components with indicators and examples together to see them in a larger picture. Each component is made more operational by using indicators. Additional examples aim to delineate the components. Some of examples presented in the table below are given in the form of the contexts/language structure to be encountered in the classroom, while others are given to explain the indicators better.

**Table 3.** Making Connections Skill, Its Indicators And Examples

| Main Component | Sub-component | Indicators | Example |
|---|---|---|---|
| **Connections between concepts** | *Making connections between the concept and others* | Using a different concept/concepts to teach another concept/mathematical statement | "**Proportion** is used to calculate the measure of **length of the arc** which is equivalent to the measure of central angle of the circle. Strictly speaking, if the length of the circumference for 360° is $2\pi r$, the measure of lenght of the arc is $\frac{2\pi r\alpha}{360}$" |
| | *Making connections between the concept and its sub-concepts and among the sub-concepts* | Using the hierarchy or the relationship between one main concept and its sub-concepts in teaching<br>Making connections between the sub-concepts of main concept | "The **equilateral triangle** is a triangle; all of whose angles equal to $60^0$and **the length of its edges** are also equal."<br>"E**quilateral triangles** are acute **triangles."** |

**Table 3.** Continue

| Main Component | Main Component | Main Component | Main Component |
|---|---|---|---|
| **Connections between different representations of the concept** | | Making connections between at least two different representations (Table-graphic, equation-graphic, verbal statement-equation, symbolic representation-picture-model-concrete object-verbal statement etc...) | "Algebraic representation of the statement; '*three times a number plus seven equals to 45*'is *3x+7=45*."<br>"Showing the unit fraction $\frac{1}{5}$ on a **pie** brought to classroom, **modeling** it on a circle and stating it as "**one fifth of a whole**" |
| **Making Connections with real life** | *Teaching the concept within real life context* | Using problems or examples including real-life contexts<br>Teaching concepts using concrete models and simulations | "What is the difference between the mean of girls' and boys' age in our classroom?"<br>"The number -10 is similar to owing someone10 liras. The number +10 is like having 10 liras in your pocket."<br>"Teaching the concept of equality using the concept of scale (concrete or simulation)" |
| | *Stating real life connection via verbal examples* | Stating the connections between the concept/statement and real life verbally | "We can see decorations made of reflexive, rotational and translation symmetric on our carpets at our homes and in Ottoman architectural work." |
| **Making Connections with different disciplines** | *Teaching the concept within a different discipline's context* | Teaching a mathematical concept/statement using one context/concept/statement belonging to a different discipline | " Introducing derivative through finding the instantaneous speed of a moving object" |
| | *Stating the connections with other disciplines by giving verbal examples* | Stating the connection between the concept and different disciplines verbally<br>Stating the use of mathematics in different disciplines only verbally. | "The concept of **rate** is used in **the field of science** to explain the concepts of **speed and density**." |

## The Relationship Between The Components Of Making Connection

Each component of making connections has been examined individually up to this point. However, more than one component may be used to solve a problem or give an example because these components are not independent of each other in practice. The following example demonstrates the close relationship between the four components to solve one problem or present a possible way to solve it.

"A car travels 2/5 of a distance of 240 km and then 2/10 of it at a speed of 80 km/h. How many hours would it travel the rest of the distance earlier at a speed of 100 km/h?"

In the process of solving this problem, it is possible to see all of the components of making connections. First of all, the concept of speed belonging to the discipline of science needs to be used. Therefore, this problem paves the way for making connections with different disciplines. The context given in this problem is related to real life as cars' speed is a context faced in real life too. In this way, the problem makes the connection with real life possible. A distance of 100 km can be modeled and then problem can be solved. This requires making connections between different representations (proper fraction and its modeling). The model regarding the solution of the problem can be divided into 1/10 unit fractions, and a connection can be made between the concepts of fraction and unit fraction. This is illustrative of making connections between concepts. Similarly, the concepts of ratios and proportions can be used to solve the problem, and thus, the fractions-ratios relationship can be established.

Another example can be given from the domain of statistics learning. Let's think that the teacher conducts a small-scale survey study about the domain of statistics in the classroom. The teacher can prepare a list to figure out what students' favorite colors are among five different colors (blue, green, red, white and yellow), and then s/he can demonstrate the data using a graphic and interpret them. This example falls in the spectrum of making connections with real life and connections between different representations of the concepts.

The examples given above enable teachers to make connections in the related component. Nevertheless, using these connections is closely related to what the teacher will teach. For example, the example about the field of statistics may focus on different aspects in the practices in classroom. This will affect which component is dominant. If the teacher pays attention to using real-life contexts, the component of *making connections with real life* will be emphasized. On the other hand, if the teacher focuses on the transformation between tables and graphics, then the component of *making connections between different representations* will come to the forefront. In a similar example given above, the concept of ratio can be emphasized, or the concept of speed or calculation of speed can be the aspect the teacher would like to emphasize. Therefore, it is teachers' discourses and examples in the classroom and authors' emphasis on the solution of the problems in the textbooks that show which component is privileged.

In conclusion, it is possible to observe the components of making connections separately or together in classroom practices, but the teacher can also make connections with other components in the context of the connections made in one component. As connections with real life and connections with different disciplines are more comprehensive, the connections regarding the other two components can be generally made as a part of them. Therefore, there is a close relationship between the components forming the content of making connections skill.

## Discussion, Conclusions and Implications

The discussion regarding making connections skill is presented in four sub-headings. Making connections skill will be firstly investigated in terms of the nature of mathematics and then of teaching and learning mathematics.

The next sub-heading will focus on the conceptual framework of making connections as a practical tool. Lastly, making connections skill will be evaluated as a life skill in a general perspective.

***1. Making Connections In Terms Of The Nature Of Mathematics***

The discipline of mathematics is hierarchical by nature; it depends on concepts both vertically and horizontally. This characteristic of mathematics reflects on the curricula, and the concepts taught over years are involved in the curricula in a way that requires a sequence or interrelationship between them. In primary school mathematics curricula, for instance, only natural numbers or cardinal numbers are included. These numbers are followed by integers, rational numbers and real numbers in the middle school curricula. At secondary school level, however, real numbers are taught in detail, and then complex numbers are taught. There is a relationship of being each other's sub-sets among these number sets, and these are presented in the curricula at different levels. Similarly, connections can be made between the concepts at the same levels. The concept of ratio-proportion can be used as a practical tool in teaching the data learning domain. The area rule of the triangles can be also used in the construction of the area rule of the rectangles. Therefore, students must construct and know vertical and horizontal relationships between the concepts to understand the connections both within the same learning domain and between different learning domains. Constructing and understanding these relationships will also lead to conceptual/relational understanding (Skemp, 1976; Hiebert & Lefevre, 1986) which has an important place in mathematics teaching. The component of making connections between concepts, one of the components of the conceptual framework regarding making connections skill proposed in the present study, aims to help students understand this relationship which is inherent in the nature of mathematics. Constructing this relationship in teaching is highly significant in developing making connections skill.

***2. Making Connections In Terms Of Mathematics Teaching***

The component of making connections between concepts of the proposed conceptual framework is most related to the nature of the discipline of mathematics, whereas components of different representations, real life and different disciplines are associated with mathematics teaching. Indeed, the connections made by using these components allow mathematical concepts to be learned at a conceptual level and make the connections between concepts more effective.

**2.1. *Making Connections Between Different Representations***

Making connections between different representations refers to learning mathematical concepts or statements at a deep and conceptual level in mathematics learning. Developing students' skills to make connections between different representations is closely associated with teacher competencies. Accordingly, Shulman (1986) proposed knowledge of different representations of the concepts to be an indicator of teachers' Pedagogical Content Knowledge. Some research indicated that teachers have difficulties about representations (Hitt, 1998). In this study, therefore, making connections between different representations is considered as a component of making connections skill and regarded as an indicator of whether teachers teach in accordance with making connections skill in their classroom practices.

Given the research conducted on different representations, it is seen that they specifically emphasize the benefits for pupil learning, but the research regarding the teacher dimension is overlooked (Ainsworth & Van Labeke 2004; Ainsworth, 2006; Alacaci, 2009; Gürbüz & Birgin, 2008; Ural, 2012; Van den Heuvel-Panhuizen, 2003). The studies conducted on how teachers use different types of representations in their classroom practices and what they do to develop students' connection skill remain few in number (Coşkun, 2013).

Different representations contribute to the development of making connections skill, yet there is a need to pay attention to the affordances and constraints of the representations in classroom practices. Teaching a concept depending on only one representation can cause this skill to develop weakly. This is an important issue argued in Ainsworth's (1999, p. 34) study. Ainsworth (1999, p. 34) argued that different representations have three main functions; *complementary roles, constrain interpretation and constructing deeper learning*. Complementary roles function means that each representation includes different information about the concepts, and in this way, the information lacking in one representation can be found in another one. For example, multiple representations (verbal, table) regarding the statement $f(x) = x^2$ will complete each other and lead the statement to be learned better.

Representations can have a constraint interpretation function as well. To give an example, even though the algebraic statement $f(x) = x^2$ includes a lot of knowledge, this type of representation limits the visibility of the aspects revealed by other representations. What is meant by constructing deep learning function is to make understanding deeper by using different types of representations together in teaching. Thus, no matter how important knowing each representation separately is, the relationship between these representations is required for a deeper understanding. This can be possible with the development of making connections skill investigated in the present study. In classroom practices, taking into consideration the functions proposed by Ainsworth (1999) and the connections between different representations can facilitate development of making connections skill. Making use of technology in this process is important in terms of development of this skill. Special attention is devoted to using technology in the studies conducted in the recent years (Ainsworth & Van Labeke, 2004; Duncan, 2010; Erbaş, 2005; Moreno & Dura'n, 2004).

### 2.2. *Making Connections With Real Life*

Mathematical concepts are abstract in nature. For this reason, students have great difficulties in learning mathematical concepts. Connections with real life are frequently used while teaching in order to help students learn mathematical concepts and encounter with less difficulty in understanding the concepts. Producing mathematics by inspiring from real-life contexts and sometimes contributing to understand the meaning of real-life contexts, specifically in mathematics taught before university education, make the connections with real life more important. Hence, mathematics' relationship with real life has a significant place in mathematics teaching and curricula (Ji, 2012). For example, Holland adopts the principles of the Realistic Mathematics Education approach and attaches great importance to real-life contexts in the mathematics teaching curriculum. Mathematics is taught in accordance with this approach (Van den Heuvel-Panhuizen, 2003, Van den Heuvel-Panhuizen & Wijers, 2005). Further, as Mosvold (2008) puts it, the curricula of most countries aim to prepare individuals for the situations they face in daily life as well as their professional and social lives in the future. This also indicates the importance of making connections with real life.

At this point, how and at what quality connections are made with real-life contexts in classroom practices is of utmost importance. In this study, it is proposed that connections with real life can be made through 'Teaching the concept within real life context' and 'Stating real life connection via verbal examples' components. Coşkun (2013) found that teachers sometimes give verbal examples from real-life contexts, but they usually ignore the component of teaching concepts within real-life contexts in their classes. Teaching concepts within real-life contexts is not difficult, and problems including real-life situations and problem-based instruction are some instruments that can be easily used within this context. However, knowing the importance of them, using them effectively and making a well-constructed connection with real life are closely associated with teacher competencies.

Although making connections with real life is suggested, it is seen that so little research is conducted on this issue in real classroom contexts. The studies conducted on this issue are mostly limited to TIMSS analyses. Mosvold (2008), for instance, compared how mathematics is connected with real-life contexts in classrooms in Japan and Holland by using TIMSS video analyses, and obtained interesting findings. Real-life contexts are seen mostly (44%) in mathematics lessons in Holland in which the principles of Realistic Mathematics Education are effective, while it happens rarely (9%) in Japan.

Hiebert et al. (2003) found in the video analysis of TIMSS that textbooks play a significant role in the lessons in Holland, and the problems depending on real life are vastly included in these books, but lessons are taught using traditional methods. However, the environments are open to discussion, students can defend their views regarding theories and operations, and they can engage in detailed discussions in the lessons in Japan. According to Mosvold (2008), the quality use of real-life contexts in classrooms is more important than their quantity. In addition, appropriate use of real-life contexts is of utmost importance as well. As discussed earlier, these are closely related to teacher competencies. In this sense, it is highly critical to develop materials about how this component can be used in teaching and to provide them for teachers.

### *2.3. Making Connections With Different Disciplines*

Work fields, objects and methods determine the identity of disciplines and make them distinctive. Nonetheless, this does not mean that they are completely separate from each other, and that there are no relationships between them. On the contrary, some close relationships emerge between different disciplines. It is often emphasized in the curricula that students must be knowledgeable about this relationship, and teachers must integrate them into their classes. For this reason, one of the components of the conceptual framework developed in this study is making connections with different disciplines.

Although making connections with different disciplines is regarded as a skill that students must acquire, there is not a clear definition of what this skill actually means in both literature and curricula. In this research, making connections with different disciplines is specifically assessed under the frameworks of 'Teaching the concept within a different discipline's context' and 'Stating the connections with other disciplines by giving verbal examples'. If the contexts and the concepts of a different discipline are used while teaching a mathematical concept, it can be said that a connection is made between mathematics and different disciplines. Furthermore, stating the use of one mathematical concept or statement in a different discipline refers to the connection with a different discipline. Therefore, mathematics and mathematical concepts are at the core of making connections with different disciplines; the concepts or contexts of other disciplines can be used as a tool for teaching and learning mathematical concepts.

In this research, making connections with different disciplines differs from making interdisciplinary connections. As Jacobs (1989) notes, interdisciplinary approach is an approach in which the contents and processes of more than one discipline are organized around a central theme, topic, problem or experience. In this approach, it can also be indicated that mathematics has a critical role in science, arts, linguistics and social sciences (NCTM, 2000). However, mathematics is not at the core here as it is in the component of making connections with different disciplines, rather there is a product or concept that can be delineated with the joint efforts of different disciplines. We can consider STEM (Science, Technology, Engineering and Mathematics) (Corlu, Capraro, & Capraro, 2014) and mathematics-science integrated curricula (Czerniak, Weber, Sandmann, & Ahern, 1999) which emerged in the last years as a part of interdisciplinary approach. In fact, interdisciplinary approach requires making connections intensively, and the proposed framework in this study can used within this approach as well. However, mathematics, mathematics curricula and mathematical concepts are taken as the central points; therefore, interdisciplinary approach is excluded from the conceptual framework in this study. What interdisciplinary approach stands for and how it can be adopted should be considered for future research.

Making connections with different disciplines should be considered in a different way from making interdisciplinary and multidisciplinary connections, and it will be useful to examine how this type of making connections can be used in mathematics teaching. This is because it is not clear how students and teachers can make connections with different disciplines as suggested in the curricula.

## **3.** ***The Conceptual Framework Of Making Connections As A Practical Tool***

Making connections skill is mostly considered as a skill for students in related literature (Carpenter et al., 2003; MoNE, 2013; NCTM, 2000). However, a conceptual framework outlining how this skill can be acquired by students in classroom practices is lacking. Moreover, the studies which focus on making connections in classroom practices (Ainsworth & Van Labeke 2004; Czerniak et al., 1999; Frykholm & Glasson, 2005; Hiebert et al., 2003; Mosvold, 2008), do not examine all of the components of making connection skill as a whole, and most of them approach this skill in terms of one component. The conceptual framework proposed in this study can serve as a guide for teachers both in the process of getting prepared for their courses and in their classroom practices. For this reason, although the framework proposed in this study is theoretical, it can still be a useful instrument for classroom practices.

**4.** ***Making Connections As A Life Skill***

Although making connections skill is considered as a mathematical process skill in this research, it is actually a life skill in general. This is because the real world requires looking at, thinking about and living many events and phenomena interrelatedly. Thus, this skill will help students get prepared for real life while developing this skill in mathematical courses. Furthermore, development of this skill can affect students' performance in other courses. Development of this skill can contribute to development of reasoning and communication skills, two of mathematical process skills, which is really vital for students' lives. All of these issues clearly highlight the importance of teaching arranged for development of making connections skill.

## Conclusions and Implications

Making connections skill is one of the fundamental skills supposed to be acquired by students. However, it is not clearly revealed what must be given to students through this skill in the related literature. In other words, what this skill really means is not identified clearly in literature. In this study, the components of making connections skill are first determined. Then the content of this skill is constructed, and a conceptual framework regarding how this skill can be acquired by students is proposed.

Further research needs to focus on evidence-based data about the functionality of this conceptual framework. Various methods and themes can be used in further research. For example, the extent to which this conceptual framework is an effective tool for teaching and learning in classroom practices can be investigated. In another words, an investigation of the influence of mathematics teaching conducted in the light of the proposed framework on students' learning can be an important further research area. In addition, given that making connections skill is one of the fundamental mathematical process skills, it may be suggested that this framework can be used as a diagnostic instrument to determine teachers' and prospective teachers' competencies regarding this skill and as an instrument for enhancing the content of professional development program.

In this study, making connection skills is examined only in terms of mathematics learning and teaching. There is a need for further research to explore whether the proposed framework is functional for determining making connection skills in such courses as science and social sciences. It may also be useful to determine to what extent, how comprehensive and with which components making connections is included in instructional materials such as textbooks, and thereby enhance the quality of learning and teaching by using this conceptual framework. Comparing mathematics textbooks in Turkey with textbooks of those countries that have shown good performances in exams like PISA and TIMSS in the light of the proposed framework can yield instructive findings. An investigation of test questions in national and international exams in the light of the framework can also yield similar findings.

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