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High School Students' Abstraction Process of the Knowledge of Signum Functions Based on Piecewise Functions

Lise Öğrencilerinin Parçalı Fonksiyon Üzerine İşaret Fonksiyonu Bilgisini Oluşturma Süreci

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Abstract

In recent years, learning environments have been radically affected by the scientific developments regarding cognitive processes. In our study, we examined high school students' process of abstraction of the knowledge of signum functions in a learning environment created by considering such developments. In the instruction phase, which employed a case study and group studies with two high school students, three specially-designed sequential problems were used. It was observed that the students used the knowledge they had abstracted for the first problem in order to solve the others, and they abstracted the knowledge of piecewise and signum functions accurately to a certain extent. The study also showed that using environmental events and problems to teach about functions strongly contributes to learning.

Keywords: Signum functions, piecewise functions, learning in context, constructing, abstraction process

Öz

Son yıllarda öğrenme ortamları, bilişsel süreçlerle ilgili bilimsel gelişmelerden ciddi şekilde etkilenmiştir. Çalışmamızda bu gelişmeler dikkate alınarak hazırlanan bir öğrenme ortamında, lise öğrencilerinin İşaret Fonksiyonu bilgisini oluşturma süreçleri incelenmiştir. Örnek olay yöntemi ve iki lise öğrencisiyle grup çalışması şeklinde yürütülen bu çalışmanın öğretim kısmında, sıralı üç problem tasarlanmış ve kullanılmıştır. Çalışmada öğrencilerin ilk problemde oluşturdukları bilgiyi, sonrakilerde de kullandıkları, Parçalı Fonksiyon ve İşaret Fonksiyonu bilgisini belirli bir düzeyde doğru olarak oluşturabildikleri gözlenmiştir. Çalışma ayrıca, fonksiyonların öğretiminde çevresel olay ve problemlerin kullanılmasının öğrenmeye olan güçlü katkısını ortaya koymuştur.

Anahtar Sözcükler: İşaret Fonksiyonu, Parçalı Fonksiyon, bağlam içinde öğrenme, yapılandırma, soyutlama süreci.

Introduction

Despite the significant developments in the last 50 years or so in the theories and applications regarding cognitive processes, the desired result has not yet been achieved regarding how mathematical knowledge is abstracted and in what ways a quality learning environment can be created (Mitcelmore and White, 2004a); Schoenfeld, 1988, 2006). Researchers working on the essence of learning have been pushed into studying knowledge abstraction process in more detail as both the Constructivist Learning Theory and Realistic Mathematics Education (RME), two of the theories that have greatly affected mathematics education, accept the notion that each indi-

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vidual abstracts knowledge him/herself.

The idea that the core of mathematical thinking involves moving from concrete to abstract (Courant, 1964; Translated by Yıldırım, 1988) emphasizes the need to study knowledge abstraction processes to succeed in teaching mathematics. Consequently, such issues like *how mathematical knowledge is abstracted* (Pape, Bell and Yetkin, 2003), what is effective in the process, and what enhances the quality of the knowledge abstracted have among the ones studied most frequently in the field of mathematics education. In the literature, such studies can be found with some keywords like abstraction, abstraction process, abstraction of knowledge and others. Elaborating on them could give us positive opportunities to organize more suitable learning environments and to teach better. Our study aims to do that and concerns the process of abstracting the knowledge of piecewise functions and signum functions as a special kind of the former, both covered by the high school mathematics curriculum in Turkey. The study is focused on signum functions and designed to contain the concept of piecewise functions as well; knowledge of the former may be built on the latter.

The execution of the study was predicated on the philosophy and principles of constructivist teaching and the definition of mathematics as "the knowledge abstracted in the problem-solving and interpretation process based on modeling the reality and the skills developing in that process" (De Corte, 2004). Considering all these reference points, the instruction was designed in a problem-based way, and the students aimed to abstract their own mathematical knowledge while working on the problems.

There is not a particular reason why signum functions were chosen as the subject matter of the study. Our principle aim was to create a learning environment where the students could work with mathematical knowledge in an abstract way, try the designed instruction and discover some clues to enhance the quality of teaching.

A group of two students were given instructions in order to carry out the study. As the study mainly investigated students' construction of their own knowledge – in other words, their process of abstracting knowledge – the concept of abstraction needs explaining here.

Abstraction is widely known as the process of passing from concrete to abstract. It was initially a concept in which knowledge theorists were interested. In time, it has become an issue that educational theorists also study. Abstraction has many different features, which is why researchers cannot agree on a particular meaning of it (Hassan and Mitchelmore, 2006). The literature has two major explanations (experimental and dialectical) of it.

The explanation known as experimental abstraction entails realizing some similarities while experiencing things and framing a concept so that it can represent the category of the secondary objects that bear those similarities. The process in which the concept is developed is called abstraction and the concept framed is known as the abstracted object. This process is remarkably similar to the process of developing such concepts as friends, colors, candor etc., which have nothing to do with mathematics (Mitchelmore and White, 2004-b). It is necessary to have more than one example so that experimental abstraction can occur. For instance, it is experimental abstraction when the common feature of four-component sets is perceived to be the number four or when it is realized that the concept of angle is basically the combination of two beams growing from the same point.

The second explanation about abstraction is the one that lays emphasis on its dialectical nature. The word "dialectical" refers to the fact that thought never stops changing and its evolution takes place as a result of inner conflicts. The dialectical explanation of abstraction is about reflecting upon a concept to reach an even more abstract form of it in every new step. This kind of abstraction does not need more than one example, but it is necessary to have wide experience in regard to the object being abstracted (Hershkowitz, Schwarz and Dreyfus, 2001). Concepts such as function, group, circle and vector space can be given as examples of that. In every step, demonstration with symbols is more than it is in the preceding level. Instead of proceeding from concrete to abstract, as in experimental abstraction, it advances from abstract to even more abstract.

Hershkowitz et al (2001), who adopt the dialectical approach, blended their first-hand experience with Davydov's (1990) theory and defined mathematical abstraction as a vertical reorganization activity of pre-acquired mathematical knowledge in order to construct a new mathematical construct. The word "activity" refers to what students do in learning environments for individual or group studies, and the phrase "a new mathematical construct" refers to mathematical thinking (concepts, correlations or generalizations) occurring as a consequence of abstraction. "Vertical organization" means having a more formal mathematical object than all the formal or informal ones in hand, working with symbols and developing relations between concepts, which is required by the second phase (horizontal mathematization is the first phase) of the mathematization process in the Realistic Mathematics Education theory (De Lange, 1996; Hauvel – Panhuizen, 1996).

As the process of abstraction cannot be observed directly (Dreyfus, 2007), it has been necessary to define some observable actions that can provide information about it. To define the process, Hershkowitz et al (2001) named the major observable epistemic actions as *recognizing, building with* and *constructing* and called their studies the RBC model using the initial letters of the abovementioned words. The main reason why these actions have been defined is the wish to have information about abstraction processes. They are all observable, which enables us to know much more about the process (Dreyfus, 2007). These considerations do not necessarily mean that every mathematical concept is produced as the result of an abstraction process. Although most of the mathematical acquisition occurs through abstracting, some knowledge and skills, like in operation algorithms, do not require abstraction. They are gained or learned by remembering and repeating things.

Dreyfus (2007) reported that the epistemic actions are intertwined and nested within each other. Although they are sometimes sequential, they can also complement each other, be the continuation of one another or take place simultaneously. Recognizing refers to an individual's knowing and/or interpreting the mathematical elements in a learning environment using his or her previous formal or informal knowledge. *Building with* means using recognized mathematical elements in ways like establishing relationships, solving problems, etc. to produce new knowledge. It is observed more precisely when students try to talk about a particular situation, defend a theorem, make a hypothesis and solve a problem as they then need to employ the constructs they learned beforehand. Constructing is the main step of an abstraction process. It is reported to be reconstructing and organizing that which is previously known with subtle changes and making new interpretations (Bikner – Ahsbahs, 2004). Constructing occurs with the contribution of the two other epistemic actions (Dreyfus, 2007) because one cannot have a new construct without carrying out the other epistemic actions with previous knowledge and experience.

The essential difference between building with and constructing is that the former is about pre-acquired (existing) constructs whereas the latter is focused on new ones. Students acquire such new constructs by solving problems and proving things. When they solve ordinary problems, they use recognizing and building with, alternately. Constructing is usually used to solve nonstandard problems, which points to the necessity of using such problems to teach target concepts.

Dreyfus (2007) reported that constructs that emerge through abstraction are brittle and this makes retaining new constructs difficult. In this respect, it is indicated that new constructs need consolidating after being abstracted and this can happen while using them to construct some others, solving different problems and pondering over them (Dreyfus, Hadas, Hershkowitz and Schwarz, 2006).

There are not many studies on abstraction processes. The existing ones are either about the processes themselves or the factors affecting them.

The study by Hershkowitz and others (2001) is one of those analyzing abstraction processes. It was carried out with a 9th grade student. The subject matter was teaching the functions and it was concluded that abstraction occurs while solving problems and using merely the actions of recognizing and building with, students fail to solve some particular problems and need help to achieve construction.

In their study on absolute value functions, Özmantar and Monaghan (2007) examined the abstraction process in an environment where communication with friends and help from the teacher were possible and found some important components of it.

Regarding the factors affecting the abstraction process, Monaghan and Özmantar (2006) investigated the process of constructing something new benefiting from existing constructs and the relations between the mathematical constructs over the functions of $y = f(x)$, $y = f(|x|)$, $y = |f(x)|$ and $y = |f(|x|)|$. They reported that a constructed mathematical object is brittle and what can be called a mathematical construct is only the consolidated form of a new construct.

In light of the RBC model, Yeşildere and Türnüklü (2008) investigated the knowledge construction processes of 8th grade students with different mathematical capacities. In the study carried out with two groups of two students with high and low mathematical capacities, the work concerned the problem of finding the relationship between the lengths of perpendiculars from a point in the base side of an isosceles triangle to the edges and the length of the perpendicular belonging to the edges. It was observed that whereas the students with high mathematical skill managed to spot their errors following the clues, the others could not discover them and complete the building.

In the problem of the absolute value function and drawing its graph in Monaghan and Özmantar's (2006) study, the students worked directly on a particular mathematical content. The study by Hershkowitz and others (2001) was based on a previously drawn function graph about the change of animal communities. Yeşildere and Türnüklü's (2008) study focused on the problem of perpendiculars on isosceles triangles. In each of these studies, the contexts in which the problems were presented had a significant mathematical content and it is quite normal that students who lack sufficient previous knowledge would not be interested in them. The context in which a problem is given is important in terms of arousing students' interest and making them think that it is worth trying to solve. Our study is about a teaching attempt designed to be suitable for recognizing piecewise and signum functions and drawing their graphs. As teaching instruments, some contexts were used with particular social values that students would find worth working on. For these reasons, the study and its results can be expected to concern the majority of students and contribute to the studies on program development and educational planning.

Method

Model

This is a qualitative study which made use of a case study method. It is a research method in which a phenomenon is examined in its own environment, the borders between the phenomenon and its environment are not totally clear and which is employed when more than one piece of evidence or source of data are existent (Şimşek & Yıldırım, 2006). Case study was used in our study as we focused on the knowledge construction steps of the students.

The case in our study comprises the students' knowledge construction, analysis units and the steps of recognizing, building with and constructing in a knowledge construction process. As is mentioned RBC theory, these steps are nested within one another. For this reason, an embedded single case design was used, in which most of the time more than one sub-strata or unit are present (Yıldırım and Şimşek, 2006).

The researcher implemented the pre-designed instructions and worked as a teacher-observer (Tsamir and Dreyfus, 2002), which is a role conceived in the literature.

Before the interviews, the permission was received from the school management and students, and the aim and scope of the study were explained to the management in detail. The students were told that the study would be recorded and it had nothing to do with their class marks and what mattered more than right or wrong answers was the extent to which appropriate answers were achieved.

The study focused on students' processes of constructing knowledge in an environment designed in consideration of constructivist learning. The investigation of the processes was based on the RBC model. Taking into account the dialectical characteristic of such processes, the actions of recognizing, building with and constructing were examined as the epistemic actions of the RBC. Data were analyzed to ascertain the extent to which the students constructed the signum function and the preliminary constructs it required.

Study Group

The study was conducted as group work with two volunteering groups of first grade students enrolled in a high school. Mathematical success of the students was not tested with a particular examination, but the school management and their math teachers were interviewed about it. The participants were reported to be successful students who could usually get between 75 and 80 out of 100 in their exams. Neither of them had been taught about piecewise or signum functions. Their knowledge about functions consisted of domains, ranges, matching, real number pairs, coordinate systems and matching between real pairs and points on planes, which are all covered in the 8^{th} and 9^{th} grade curricula.

Qualitative data collection methods such as observation, interviews and document analysis were used. A video camera was used in collecting the data.

As is the case in the studies by Dreyfus and others (2006), Hershkowitz and others (2001), Özmantar and Monaghan (2007), and Yeşildere and Türnüklü (2008), our study was carried out with two students at the same time and place. The aim was to make them think out loud by giving them the opportunity to have peer help and discuss. At the beginning of the study, some questions and explanations were presented to acquaint the students with the contexts of the problems. Whenever needed during the solving processes, new questions were asked in order to obtain the students' opinions and observe their verbal and nonverbal communication with each other and the researchers. Next, all the image and voice recordings were analyzed in terms of the principles of constructivist learning such as learning involves active cognitive processing, learning is adaptive, learning is subjective, not objective and learning involves both social/ cultural and individual processes (Doolittle, 1999).

While one of the researchers observed and recorded the action, the other one worked as the teacher-observer.

The data were derived from the sheets on which the students solved the problems, the observations of the researchers and the video recordings made during the interviews.

Data Collection Instruments

Pilot study: The questions were asked of two students with similar levels of success in the classroom where the study was carried out; some amendments were made in the texts of the questions. It was observed that the students had advanced knowledge (e.g., coordinate system, doubles of real numbers) to solve questions, were impressed by the questions and were able to discuss them with the researchers.

The data collection instruments were the worksheets containing the solutions of the problems used in the case study. The first two problems were on piecewise functions and the third was about signum functions. In order to construct preliminary knowledge about signum functions, the target of the study, two problems were first chosen about events suitable for piecewise functions and then another one that required drawing the graph of the signum function. The problems are given below in the order they were used in the study.

The following are the costs of specific lengths of computer use in an internet café.

50 kuruş up to 30 minutes,

75 kuruş for any duration of use between 30 and 45 minutes

and 100 kuruş for any duration of use between 45 and 60 minutes and 25 kuruş more for each ensuing use of 15 minutes.

- A student uses a computer for 17.5 minutes on the first day and 48 minutes on the second. How much is he to pay for each day?
- Draw a graph so that every user can infer the amount of money to be paid for any length of computer use.

1. For sales promotion, a store gives its customers gift vouchers according to costs. The minimum amount you need to spend to get one is 50 liras. The chart below is about the number of gift vouchers to be given.

1 for a cost between 50 and 150 liras,

2 for a cost between 150 and 300 liras,

3 for a cost between 300 and 500 liras,

4 for a cost between 500 and 1000 liras

and 5 for a cost between 1000 liras and more.

Draw a graph so that customers can infer how many gift vouchers are given according to any amount of money spent.

2. Shooters firing at the rings nested within each other on a target board get 0 points if they hit the ring numbered 0, -1 point if their bullets go wide and +1 point if they hit the inner rings. The points won are shown on a scoreboard right after the shots.

- What can be the rings hit by a shooter who gets the points of -1, +1, 0, -1 and +1 respectively with his 5 successive shots? Give two different examples.
- Draw a graph of points (shots) to show the points to be won if the shots were too many, even limitless.
- Design and draw a target board with rings as thin as a compact disc.

In choosing the problems, two specific features were taken into consideration. First, in accordance with Dooley (2007) and Dreyfus's (2007) studies reporting that nonstandard problems are more appropriate for constructing new mathematical constructs, nonstandard problems were chosen. The other feature sought in the choice of problems so that abstraction could occur was self-developing models that function as a bridge between informal and formal mathematical knowledge, which is a principle in Realistic Mathematics Education (Gravemeijer, 1994). Many different problems can be chosen to form models for teaching target constructs, but the best one(s) should be preferred. The following components were also taken into consideration:

(*i*) a dialectical environment appropriate for students' levels of development, (*ii*) the presence of a thing to be abstracted, (*iii*) the need for a teacher's help or guidance in mathematical interpretation, which Özmantar and Monaghan (2007) deem effective for the occurrence of abstraction.

Each of the problems was given to the students printed on an A4 sheet one after the other in the order mentioned in the findings section.

Data Analysis

The data were descriptively analyzed. As the study was based on the RBC model and in order to precisely observe the epistemic actions of recognizing, building with and constructing, the voice and image recordings were converted into texts. Next, the observation notes taken during the implementation were assessed. Finally, some comments were made in order to interpret the findings, explain the relationships between them and draw conclusions.

The Validity and Reliability of the Study

The study was designed within the framework of the relevant theoretical literature. The results obtained at the end of the study and the concepts and content taken into consideration formed a meaningful unity altogether. The findings are in coherence with the theory and in parallelism with former studies in the field. The data were collected with a deeply-focused method in a long term face-to-face interaction. The authors intervened in the process with probing questions when the students had difficulties and when more detailed information was to be asked from them. In the end, the basic results of the study were shared with the students so that internal validity was wholly provided through participant verification. The external validity was provided with reporting in detail what had been done throughout the process. Descriptive information about the students participating and the relevant dialogues with them in the process were all reported in detail. How the researchers got the results was shown by giving, under relevant student statements, the deductions about which step of the knowledge construction process the students were in.

To provide the reliability of the study, the recorded dialogues with the students and the researchers' observation notes were coded by two different experts and interpreted in terms of the observability of epistemic actions. Besides, the statements of the students were most of the time expressed directly within the text of the study. The data sources were described in detail. It is thought that this can be guiding for other researchers in determining their data sources when carrying out similar studies. The raw data of the study were kept so that others can examine them later.

Findings

The students' processes of constructing the knowledge about piecewise and signum functions are described below in consideration of the epistemic actions of recognizing, building with and constructing (M: Mehmet, T: Tuna [these are not the real names of the students], R: Researcher).

Mehmet and Tuna spent 12, 10 and 8 minutes on the first, second and third problems respectively, which is a total of 30 minutes.

The Analysis of the Process

The analysis of the learning process is given below with emphasis on the situations when recognizing, building with and constructing behavior was observed.

The Internet Café Problem (Figure 1)

The researcher gave the first sheet containing the internet café problem and asked the students to read it and addressed some questions about the context of the problem to facilitate grasping it. The conversation concerning that part is given below.

100[***] R: Have you ever been to an internet café?

101 M-T: Yes.

102 R: The problem on this sheet is about paying the bill in an internet café. Read it.

Mehmet read it aloud and Tuna listened to him attentively.

103 R: (focusing on the first item of the problem) Can you understand the tariff Mehmet?

*** The numbering in the conversations is started from 100 to have homogeneity in the numbers of the lines.

How much does a person have to pay if he uses the internet for 17.5 minutes according to that tariff?

104 M: According to the tariff?

105 R: Yes, imagine that you went to that café and used a computer for 17.5 minutes. How much would they charge you according to the tariff? Tuna, you also can give the answer.

106 T: Can I make calculations?

…

116 M: It must be paid 50 kuruş (100 kuruş are equal to 1 lira in Turkey) for 17.5 minutes.

117 R: How much must be paid for the second day?

118 T: 100 kuruş.

It was observed that the students knew the concepts of the internet and time limits in the context and understood the problem, but had some difficulties grasping the concept of fees. The reason could be that they thought that normally much less money is paid for shorter use than for longer use, which is a consideration stemming from their knowledge of ratios and proportions. This possibility is verified by the attempt to calculate the fee with proportions (106T). The students' answers to the researcher's other questions about the fee (116M, 117R, 118T) suggest that they understood the problem. In the following phase of the study, the students were provided the second item of the problem and asked to draw a time-fee graph. As is seen in Figure 1, the names of the axes were given in the system onto which the graph was to be drawn, and the students were asked to write the values only.

As the conversation below shows, the students had no difficulty in writing the values on the axes (129M, 130R, 131M).

125 M: Here are 30 minutes and 45 minutes, so 50 kuruş for 30 minutes.

126 R: And then?

127 M: 75 kuruş for 45 minutes.

128 A: Fill in the fee axis. What do you need to write? What you have done is true, but what should follow 50 and 75?

129 M: 100.

130 R: And then?

131 M: 125, 150.

…

149 R: Now draw a graph that can show a suitable price for any duration of use.

150 M: Any… I can draw something straight here (drew the graph in Figure 1).

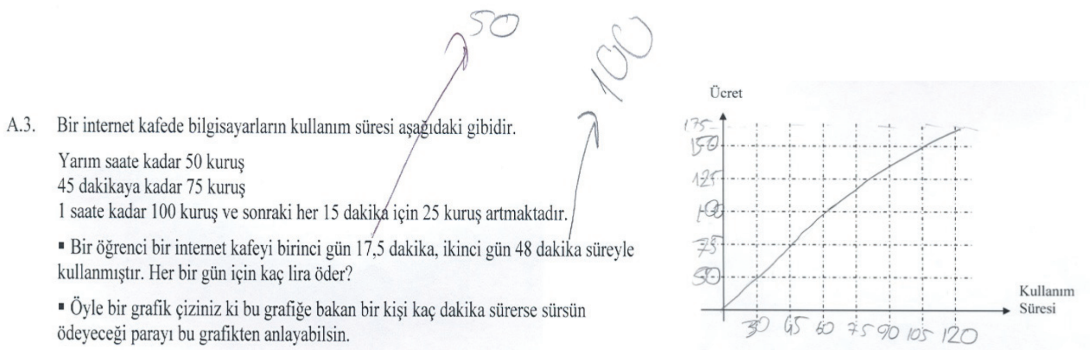

Figure 1. The First Graph Mehmet Drew about the Internet Café Problem

Although they discussed the topic of the problem beforehand and agreed with each other, it was observed that they drew the graph as a continuous line rising proportionally (Figure 1), which is a consequence of the constructs they had formed about the concept of graphs. The researcher's questions in the conversations below caused contradictions between their considerations.

151 R: Let's test if this graph provides an answer to the question. For example, how much would they charge for 35 minutes?

152 M: 65 kuruş (the students seemed to get the feeling that the drawing did not provide the answer).

153 T: How can we do it? How about this? (drew the steps) (Figure 2).

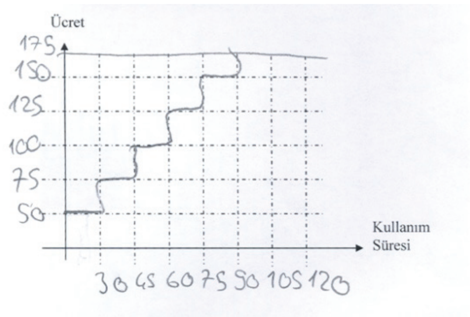

Figure 2. The First Graph Tuna Drew for the Internet Cafe Problem

154 R: It looks like stairs. Let's analyze together if this graph is true.

155 T: 50 kuruş if used at any duration until it is for 30 minutes, 50 kuruş for 21 minutes.

The researcher's queries about if the drawings provided any answer made the students convert the graph into steps (Figure 2), which shows that they realized the fact that same prices had to be due for specific time spans of use.

156 R: Do these steps have to be connected? What are the connections for?

157 T: Oh, yes (erased).

After the question "Do these steps have to be connected?", Tuna immediately began to erase saying, "Oh, yes"; this shows that he was not sure about connecting the lines. They managed to draw the graph correctly without losing any more time (Figure 3).

The students seemed to feel relieved to have found the correct drawing. However, the graph still had a major deficiency: the ambiguity in the pieces to which the points of disconnection belonged. The students were not aware of this. They were asked the following questions to help them.

158 R: 50 kuruş for any length of use until 30 minutes, and 75 kuruş for until 1 hour… If a customer used a computer for exactly 30 minutes, would you charge him 50 or 75 kuruş? What fee does this graph show?

159 T: Yes, there is something missing there.

…

164 R: Read the sentence again. Decide on what the word "until" means in the clause "until 30 minutes".

165 T: 30 minutes … Until… Before…

166 R: Is 30 minutes included in that fee, or not?

167 T: It is … No, not.

168 A: OK, keep that part blank then; you need to get to the upper level if half an hour is fully completed; now you can carry on.

T: (drew the graph specifying the points as filled in and blank).

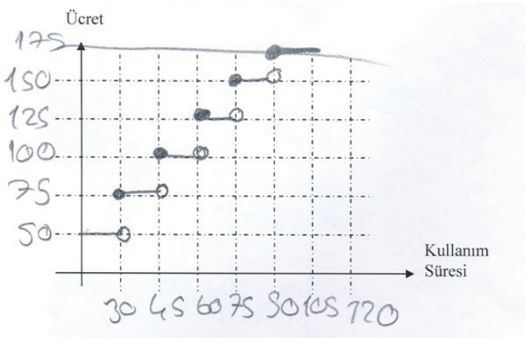

Figure 3. The Graph Mehmet and Tuna Drew

169 R: This is a very successful drawing; let's give it a name… Have a look at your previous drawings, why did you think in different ways? Can you explain it?

170 M: First, we did in the classic way we had been taught in classes.

This part of the study helped the students conclude that the left tips of the true pieces belonged to the graph, and they specified the included points as filled in and the ones not included as blank (Figure 3). They attributed their errors in the first two graphs to their habits in classes (170M).

The Voucher Problem (Figure 4)

The second question, called "the voucher problem," was also about piecewise functions. Before reading the problem to the students, questions were asked to introduce the context and give clues about the problem. The conversation below contains those questions.

200 R: You know the Korupark shopping center, right?

201 M-T: Yes.

202 R: There is a special offer there, with the slogan of "spend 50 liras, have a gift voucher of 40 liras". The problem is about that. Read it Mehmet.

M: (read it aloud).

203 R: I did shopping there the other day, it cost 148 liras, how many vouchers do you think I got?

204 T: 1.

205 R: How many would I have got if I had spent 240 liras?

206 T: 3 vouchers.

207 R: Why do you think so?

208 T: Oh, no, 2 vouchers!

The students were able to tell the number of vouchers to be received according to the costs, which suggests that they understood the problem. This shows that their experience with the fees in the first problem worked, and they used the recognizing and building with actions.

211 R: Now draw the graph of this so that any person looking at it can see how many vouchers he is to get according to the money he spends.

212 M: I'll write the prices on the horizontal axis. 50,150…

213 R: Think now, should it be 50,150 or 50,100?

214 T: 50, 100, 150…

215 R: What are you going to write on the vertical axis?

216 M: The numbers of the coupons.

217 T: I'll do it (he wrote). Is it enough if I write up to 5?

218 R: It is, now draw the graph.

219 T: The same again… Something similar will come out.

The students correctly answered the questions about understanding the problem (204T, 206T…) and proved that they understood it. This understanding is also verified by what they marked on the axes.

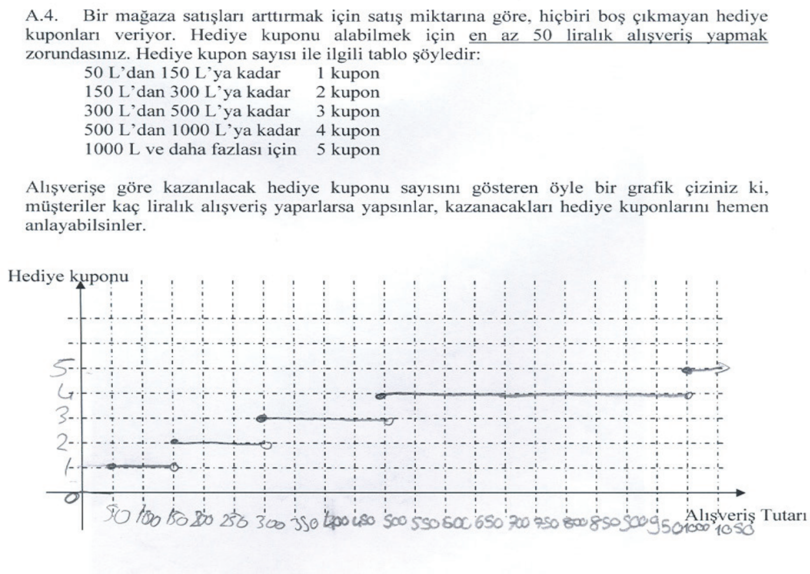

Figure 4. The Coupon Numbers – Shopping Costs Graph Mehmet and Tuna Drew

Additionally, they used what they had constructed while working on the internet cafe problem and realized that they would have a piecewise function (219T).

They had some hesitation in drawing the 0 - 50 range and stated that they decided to keep that part blank. Following the researcher's advice, they had the idea that the part corresponding to 0 vouchers could be drawn and have an arrow placed onto the right end of the last part of the graph so that it could demonstrate the costs higher than 1000 liras. They looked determined and self-confident while they were deciding on the ends to be filled in and left blank.

230 R: Which one is to be filled in?

231 M: The upper part (meant the line at the top and filled in the left ends of the line pieces).

In determining the pieces to which the points of disconnection belonged, the constructs that emerged for the internet café problem worked, and the students had the accurate drawing without any hesitation while filling in the bottom ends of the line pieces.

The Target Board Problem (Figure 5)

The ultimate goal of this study was to teach the students signum function graphs. The abovementioned problems were presented to create the preliminary constructs required by signum functions. Next, the students were asked to read and solve the target board problem. As was the case with the previous problems, the first item of the problem aimed to help them grasp the problem itself and its context. The conversation about it is below.

300 M: (read it aloud).

301 R: Tuna, you give an example of a point of shot and Mehmet, you give another one (the researcher demanded examples about the first item).

302 T: (They marked their examples on the sheet where the problem was).

303 R: For instance, can you show me with your pencil where a person with -1 point could have shot?

304 M: Out (pointed with his finger but hesitated to mark it with pencil).

305 R: Show the point of the shot… Actually, you showed but didn't mark it.

306 M: (marked the point he had shown before)

307 R: OK, where could a person with +1 point have shot?

308 M: (marked it correctly)

309 R: A person with 0 point?

310 M: (correctly marked a different place corresponding to 1)

311 R: Very good, what about a person with -1 point? Is that ring absolutely necessary to be shot then?

312 M: (marked correctly).

313 R: Tuna, you take the shots now.

314 T: Mine will be like Mehmet's.

315 R: No problem, where could a person with -1 point have shot?

316 T: (pointed at a place adjacent to Mehmet's shot).

317 R: Is another place not possible for getting -1?

318 T: It is, here, for example (marked the last one of the outer rings).

319 R: Determine a place to get +1 and one to get 0.

320 M: (nodded his head in order to show his approval to what he had just marked).

321 R: Find a place to get -1, and then one to get +1 again.

The students seemed to have understood the problem but had some hesitation to mark their answers on the target board (303R, 304M). After Mehmet made his own marking, Tuna was channeled into making his beside Mehmet's (313R, 314T, 315R, 316T), and with the researcher's guidance (317R), the students correctly located different places to get the same points. After the researcher made sure that the problem was grasped, the conversation about the second item took place.

325 M: (read the second item of the problem)

326 R: The rings are quite many and within one another, can you add any extra rings in the target board?

327 M: (drew) (Figure 5).

328 R: Are you going to write any numbers in those rings?

329 M: (wrote -4) (Figure 5).

B.1. Şekildeki hedef tahtasına ateş eden bir kişi; iç içe çizilmiş halkalardan 0 numaralı alana isabet ettirirse 0, dışına isabet ettirirse -1, iç halkalara isabet ettirirse +1 puan alıyor. Atışların arkasından, alınan puan elektronik bir tabloda görüntüleniyor.

- Yaptığı 5 atıştan sırasıyla ; -1, +1, 0, -1, +1 puanlarını alan bir sporcunun isabet ettirdiği halkalar neler olabilir? İki farklı örnek veriniz.
- Atış sayısının çok fazla, hatta sonsuz olması durumunda atışlardan alınacak puanları gösteren bir "puan (atış) grafiği" çiziniz.
- Şimdi bu hedef tahtasının bir bilgisayar cdsi gibi çok ince halkalara sahip olduğunu tasarlayınız. Bu durumda grafiğin nasıl bir hal alacağını çizerek gösteriniz.

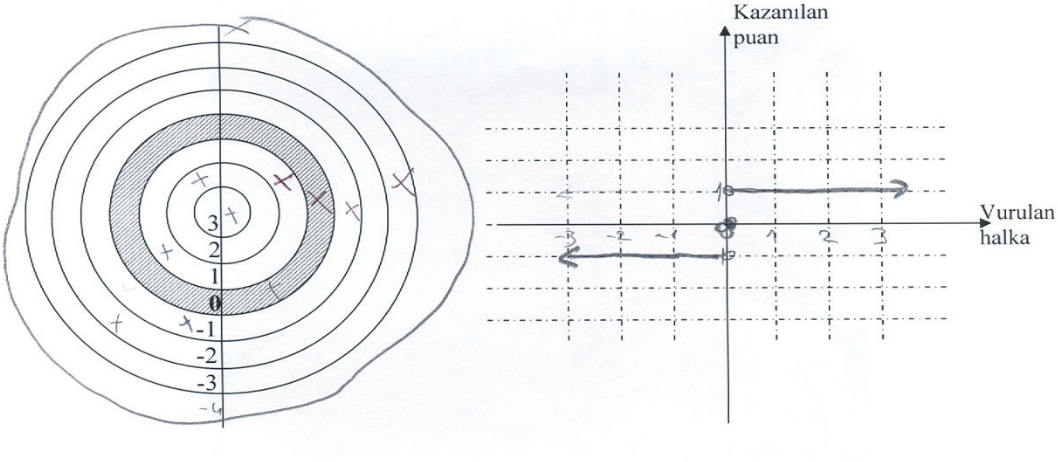

Figure 5. The Markings and the Graph the Students Made for the Target Board Problem

330 R: So there is an incessant shooting. Draw such a graphic that can show all the points won in the end. What do you need to do? Read the third item as well.

331 M: (Read it)

332 R: It should continue this way, let's come to the graph, name the axes first.

333 M: The ring shot, +1,+2,+3,-1,-2,-3.

334 R: You didn't place the 0.

335 T: (marked the 0)

336 M: And then the point won, +1, -1.

In this item of the problem, the students were asked to add a new ring to the target board. As is seen in Figure 5, Mehmet drew the new ring correctly and wrote "-4" on it. After this behavior, resulting from recognizing the target board, the researcher asked the students to read the third item and tried to understand if they realized the fact that the rings were not limited with integers only and could get more to include intermediate values as well. Proposing the values of "+1, 0 and -1" for any shot concerning integers or decimals, they proved that they grasped the new form of the problem as well. Their learning regarding the drawing phase can be followed from the conversation below.

337 R: Your answers have been true. Now draw the graph, please.

338 M: This part should be like this (drew the y = +1 line parallel to the x axis).

339 R: So what is the result, everybody has +1, right?

340 M: Oh, they are to get -1 (erased what he drew and made the correct drawing)!

341 R: It seems Tuna will draw the other part.

342 T: (drew it).

Mehmet drew the y=+1 line and showed that he understood the problem even though he did not draw it accurately. After the researcher's question (339R) "So what is the result, everybody has +1, right?", Mehmet revised and corrected his drawing.

345 R: Yes, what about the ring of zero here, what point is someone shooting on the diagonally lined ring to win?

346 M: 0.

347 R: Ok, but can we infer this from the graph?

348 M: Yes.

349 R: How? I can't do it.

350 M: Here… it's blank.

351 T: Are we to fill in this part (showed the 0.0 point)? When that is filled in and this part is blank (showed the 0.1 point), it is to be given 0.

352 R: Great… This is a signum function; different parts of it have the values of -1, +1 and 0.

The researcher's question about how 0 (zero) is gotten can be seen (345R, 346M, 347R) drew the students' attention to the point of origin and impelled them to filled it in (351T). Marking as blank the (0,1) and (0,-1) points above and below the (0,0), they employed the preliminary constructs of piecewise functions about limit values (Figure 5). Eventually, the researcher asked the students to compare the graph they drew for the last problem with the ones drawn for the previous problems and wanted to know if they could see any difference. The following conversation took place.

As is evident from the conversation above (353R… 357M), the students realized the piecewise functions and their knowledge of the concept of functions and graphs took a new form. When they were asked to explain the difference between the last problem and the others, they contented themselves with saying that the last function had three parts.

Discussion and Results

The major aim of this study was to propose, implement and discuss a model that could work for teaching signum functions to high school students. Two students who had never studied signum functions before were given three specially prepared problems in a specific order. Next, we investigated to what extent the students constructed the concept of signum functions.

The major points were:

(i) teaching by means of real problems,

(ii) giving students opportunities to construct (abstract) knowledge themselves

(iii) having positive results in terms of quality teaching.

Taking these into consideration, the study can be assessed as follows:

The Use of Real Problems in Teaching

This study was predicated on problem-solving-based instruction and the students became interested in all three problems and made an effort to solve them.

It can be said that the conditions needed for acquiring mathematical knowledge through modeling reality (De Corte, 2004) were provided to some extent. Although the internet café and voucher problems were not about real situations, they were inspired by real life. In order to model reality, mathematization had to be the focal point (Gravemeijer, 1990), and, in the internet café problem for example, the students were to feel the need to prepare an operational fee, even if they were not given one as preliminary knowledge, and determine the piecewise function to draw its graph. Considering the fact that modeling reality is not always an easy thing to do, problems of that kind can be used to create mathematization processes. As a matter of fact, the students managed to construct the mathematical knowledge working on the two problems given, which can be seen in Figure 3 and Figure 4.

On the other hand, the third problem can be considered a reflection of social reality rather than a physical one. The reason is that winning -1, 0 or 1 point with the rings shot on the target board was completely designed by us and not a requirement of any physical situation. The first two problems, which necessitated modeling the physical reality and corresponded to the concept of piecewise functions, were given to serve as a basis on which the students could build the knowledge of signum functions. The instruction phase justified doing that, and the students did not hesitate to use the nominated values while solving the third problem. The inclination to do no more than draw the y=1 line (339R) in the target board (third) problem can be explained with the difficulty of constructing the mathematical model of social realities. However, it is a fact that the second and third problems were solved in a shorter time than the first one (10m, 8m), and the students were seen to have gradually increasing motivation and enthusiasm. The instructional conversations about the problems show that the students constructed knowledge while having discussions with each other. These findings suggest that the problems functioned as an effective means of reaching the target mathematical knowledge.

The Realization of Abstraction

In this study, some constructs about piecewise and signum functions emerged in accordance with the order of the problems.

In all the problems, some right and wrong uses of recognizing, building with and constructing were observed, which are the observable epistemic actions of abstraction processes (Dreyfus, 2007).

In the first problem, examples of wrong use of the preliminary constructs include the students trying to calculate the fee using direct proportion (106T) and drawing a graph with a regular upward curve (Figure 1) although they had properly understood the text of the problem (104M, 118T). This can be attributed to the monotony of the problems used in traditional teaching. The fact that they designed the graph required by the problem as a set of connected steps (Figure 2) may be because they were affected by their preliminary knowledge still conditioning them to think that function graphs are continuous. However, they managed to accurately draw the graph shortly afterwards (Figure 3), which indicates that a new construct emerged (157T) to show them that functions can also be piecewise. Thanks to another construct that emerged, they realized the fact that a point belonging to a function might be in only one of the parts making the graph (164R, …167T).

The second (voucher) problem was presented in order to make the students use the constructs that emerged while working on the first problem and consolidate them. About the second problem, Tuna's words, "The same again… Something similar will come out" (219T) show that they were able to employ the construct that emerged in the first problem and thought that function graphs can also be piecewise. Another proof of the emergence of the new construct is that they realized the possibility of charging the same price for the values in a particular range with regard to the fee (155T).

R*ecognizing* and *building with* actions used while solving the target board problem (326M, 328M, 329M, 333M, 335T) shows that the preliminary constructs constructed during the previous problems needed to be employed for the third problem. The students drew accurately and in a shorter time the signum function, although it contained nominal values. This study followed the route of teaching signum functions benefiting from knowledge of piecewise functions or, in a general sense, teaching something designed benefiting from what can already be observed.

Upon the researcher's request for the comparison of the properly completed signum function graph with the others, the students mentioned that it was also a piecewise function consisting of only three parts and failed to see the emergence of a different function group. This provoked the thought that extra work was needed that requires drawing signum functions for abstracting them experimentally. It can be a new research subject if extra work in that way can really help high school students think that such a group of functions exist. Failing to do that, our study demonstrated that there is a need for studies on experimental abstraction.

This study also showed that recognizing, building with and constructing, which are known as the epistemic actions of abstraction, are not linear (Dreyfus, 2007; Yeşildere and Türnüklü, 2008), but nested within each other. While trying to solve the internet café problem, the students *recognized* the axes and the coordinate system before setting out to draw the graph, and they placed the right values onto the axes (125M…143R), which is an action based on a previous *building with* action.

Results in Regard to Teaching

This study suggests that using facts and/or realities as a base can enhance the quality of teaching. Finding a natural model is not always easy to do, and it is almost impossible to find one corresponding especially to all the functions that include nominated values. It is not the way a natural event looks that clarifies what makes a signum function and denominates some in -1; it is more of an assignment. The piecewise function problems used in this study were effective in teaching signum functions with nominal values, and this led to the emergence of the constructs necessary to teach about signum functions, which indicates that students need to first work on functions in which natural events are reflected so that they can interpret more complicated assignments. Building the concept of signum functions on that of piecewise functions, this study highlights the importance of the order of topics to be taught. Similar studies can identify the need to revise the order of topics taught at high schools, according to which linear functions are followed by quadratic, trigonometric, periodic, exponential, logarithmic, signum, greatest integer and piecewise functions (Ministry of National Education, 2008). Another result of the study is that the use of real and nonstandard problems can contribute to the acquisition of more quality mathematical knowledge. Choosing suitable problems in program development can be an important factor in achieving goals. Instead of the traditional way of teaching based on giving definitions first, using real events and problems and helping students mathematize them can make it easier to form concepts and consolidate previous constructs. On the other hand, as Dooley (2007) reports, since this way of teaching exemplified in our study is shaped by students' way of thinking, many researchers have preferred group work instead of classroom implementation.. As learning environments are classroom-based in Turkey, instruction predicated on such knowledge construction processes can become widespread only in classroom or studies with big groups. This is the reason for the suggestion produced in the study.

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