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# The Role of the Teacher in Supporting Imagery in Understanding Integers

# Tamsayıların Anlaşılmasında Öğretmenin İmgelemeyi Desteklemedeki Rolü

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*Abstract*

This paper presents the results of a design experiment conducted in a 7th grade mathematics classroom aimed at improving students' understanding of integer concepts and operations. The study particularly focuses on an expert teacher's role in helping students develop meaningful imagery which students can use as a foundation to fold back and rely on as they engage in further mathematical activities. Toulmin's model of argumentation is used as an analytical tool to document when an image becomes taken-as-shared by the classroom community. The results suggest that the practices of the teacher played an important role in students' development of various images in understanding and solving integer problems meaningfully as well as communicating their ideas effectively.

*Keywords:* Imagery, teacher education, mathematics education, teacher practices

*Öz*

Bu çalışma, yedinci sınıf matematik öğrencilerinin tamsayı kavramı ve işlemlerini daha iyi anlayabilmeleri için yapılan bir araştırmanın sonuçlarını sunmaktadır. Çalışmada özellikle uzman bir öğretmenin öğrencilerin yeni matematiksel kavramları anlamalarında geriye dönük olarak kullanabilecekleri mantıklı imgelemeyi geliştirmelerindeki rolü araştırılmıştır. Çalışmada Toulmin tartışma modeli öğrencilerin oluşturdukları imgelerin tüm sınıf tarafından kabul edilip ortak olarak kullanılıp kullanılmadığını analiz etmek amacıyla kullanılmıştır. Sonuçlar, öğretmenin kullanmış olduğu yöntemlerin öğrencilerin tamsayı problemlerini anlaması ve doğru çözmesinde olduğu kadar, fikirlerini iletmede de etkili olan imgelemelerin gelişiminde önemli bir rol oynadığını göstermiştir.

*Anahtar Sözcükler:* İmgeleme, öğretmen eğitimi, matematik eğitimi, öğretmen uygulamaları

## Introduction

Research studies indicate that students have constructive resources to find out different ways of solving a novel problem (diSessa & Sherin, 2000; Schwartz & Martin, 2004). However, research also emphasizes that the teacher has an essential role in helping students generate meaningful mathematical solutions and different representations as one cannot expect students to discover all different methods (Kirschner, Sweller, & Clark, 2006). One way to help students ground their mathematical activity with meaningful "anchors" that serve as the basis for more sophisticated, abstract mathematical reasoning is by supporting imagery (Thompson, 1996). Focusing on students' imagery can create opportunities for class discussion in which students can develop meaningful solution procedures which are likely

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to be remembered and productively used in future activities.

Imagery can be defined as “giving meaning to the definition through producing a highly refined image that supports the formal arguments” (Pinto & Tall, 1999). It plays a crucial role in the development of abstraction and generalization (Presmeg, 1992). Supporting meaningful imagery helps students create meaning for mathematical activities and adds richness to their mathematical reasoning and understanding (Thompson, 1996). Imagery can encompass various forms such as *personal images*, *background images*, *notational images*, and *situation specific images*. Personal and background images include those forms of imagery that the students have accumulated through their personal experience and previous classroom instruction. Notational imagery refers to what a person perceives from the given tools and formalizations. Situation-specific imagery, on the other hand, involves mental representations conjured up in a person’s mind within a given situation or context. In these definitions, tools correspond to resources that students can use to construct and communicate their mathematical reasoning and develop notational imagery (Confrey, 1990). The context constitutes a setting where students can ground their mathematical activity to develop situation specific imagery (McClain & Cobb, 1998). These four forms of imagery are related with each other in that students can connect situation-specific and notational imagery with their background and personal images. In this article, our primary interests are in both situation-specific as well as notational imagery as students can develop these through use of a supportive context and appropriate tools.

Imagery develops progressively rather than forming immediately. Pirie and Kieren (1989) emphasize the importance of supporting imagery with a recursive theory where the effective actions function with the initiation of imagery. The first level of the theory includes images that are based on situation specific activity which constitutes the initial step for abstraction. The researchers state that “it is the learner who makes this abstraction by recursively building on images based in action” (p. 8). The initial development of students’ thinking starts with taking the images that they created before (without creating a new one each time) and directing their thinking in order to make sense of the subsequent task. Mathematical development continues as students reason by using or adapting the images’ properties and as they engage in more instructional tasks that start to formalize these properties (McClain & Cobb, 1998). By folding back to the initial image students can create new meaning for the objects they are working with and move to more sophisticated levels of thinking. Here it is important to note that students use or transform their initial images to make sense of more complicated ideas. Thus, when those images are imposed from outside or given ready-made to students without regard for students’ own images, this may not contribute to development of their mathematical reasoning (Pirie & Kieran, 1989).

Sfard’s (1991) theory also describes students’ way of thinking as *operational* and *structural*. While operational thinking refers to conceiving a mathematical entity as a product of a certain process, structural thinking in general evolves from operational thinking and refers to conceiving a mathematical entity as an object. For example, non-pictorial mental images are more relevant to operational thinking and pictorial mental images are mostly related to structural thinking. One of the important roles of the teacher is to help students make transition from operational thinking to structural thinking. Here it is important to note that these concepts are complementary rather than exclusive. By using an instructional design where students can develop meaningful images associated with the mathematical objects, the teacher can prevent students to create incorrect images that may cause difficulties in solving future problems. In this study, the teacher supported students’ understanding of integers by creating an instructional sequence where students could develop meaningful images as they made sense of integers and integer operations.

*Literature Review*

*Integers*

Many studies have investigated the ways that improve students' understanding of integers. These studies used different models which can be categorized mainly in two groups: *neutralization model* and *number line model*. While the neutralization models use physical objects such as colored chips or tiles to represent positive and negative numbers and show the operations by manipulating them, number line models represent the operation by the direction of movement along the line and the numbers located based on its position and the distance (Lytle, 1994).

There have been many studies that used neutralization models. While some of these studies used abacus as a concrete model for representing operations with integers (Dirks, 1984; Linchevski & Williams, 1999), others used algebra tiles to represent positive and negative numbers with differently colored squares (Maccini & Ruhl, 2000), models based on collections of electromagnetic charges (Battista, 1983) and a helium-filled balloon model (Janvier,1983). For the number line model Thompson and Dreyfus (1988) used computer microworld to teach integers. The studies comparing these models showed that the students who used neutralization models were slightly better than the other for addition problems. However, both groups had difficulty in subtraction problems especially the ones that include different signs (Lytle, 1994). In another comparative study, Hayes found a similar result in that students who learned integers using integer tiles (a form of algebra tiles) performed better than those who learned the subject using the number line model (1999). However, she also found that these two groups of students performed similarly when the comparison was made several months after the instruction. She attributed this to a subsequent instruction that did not maximize the benefits of the improved operational skills obtained by using integer tiles.

In another study, Kinach (2002) evaluated preservice teachers' understanding of integers concepts and operations. She observed that many preservice teachers hold an instrumentalist view of mathematics, and therefore they cannot justify the meaning of the procedures that they are carrying out. To address this, she designed a methods course where she used both neutralization and number line models to support the teachers' pedagogical content knowledge of integers. The results indicated that effective use of both models create a greater understanding of the integer concepts and operations, and promote relational views in place of instrumentalist ones.

Additionally, there have been also studies that used debt (negative) and asset (positive) concepts in story problems to support calculations with integers. While debt and asset models share similarities with the neutralization models, they can be more appropriately considered as a separate category in that they involve abstract entities while the neutralization models involve concrete ones. In a previous study, when the researchers compared the students' performance that used story problems involving debts and assets with those that were asked to solve formal equations, the former group of students' outperformed the latter. However, the study also emphasized that using story problems did not enhance students' mathematical performance especially when the story cues were misleading (Mukhopadhyay, Resnick, & Schauble, 1990).

The expert teacher in this article also created and used a model based on assets and debts using the theory of realistic mathematics education (Gravemeijer, 1994). RME considers the teaching and learning of mathematics as both a social and individual activity, where students not only learn as they work on problems individually but also as they engage in fruitful mathematical conversations (Cobb & McClain, 2001). An essential component of RME is that the instruction should be experientially real in that students should engage in personally meaningful activities. This allows students to use their informal knowledge of mathematics grounded in situation-specific imagery as a starting point in developing progressively more formal mathematical reasoning that is also supported with notational imagery. To support students' imagery, the teacher used the recursive theory (Pirie & Kieran, 1989), where students first built on their informal knowledge of owing and owning, and then developed progressively more formal abstractions. This study aims to investigate the teacher's role in supporting students' imagery in teaching integers with this model that was created by the teacher.

*Emergent Perspective*

The theoretical framework that guides this study is the emergent perspective which combines a constructivist psychological perspective with an interactionist social perspective (Cobb & Yackel, 1996; Stephan, 2003). From this perspective, learning is viewed as occurring both in the individual and social realm simultaneously; one does not take precedence over the other. From this view, an individual's cognitive reorganizations constitute an act of participation in the social context of the classroom and equally, the mathematical practices of the classroom embody the taken-as-shared mathematical activity within which individual growth occurs. Therefore, as individual students create their own personally meaningful imagery for integer concepts and operations, their contributions to classroom discourse are treated as acts that help create the taken-as-shared imagery of the classroom.

Simon (1997) examined teachers' role from the emergent perspective and found that, when viewed from a cognitive lens, the teacher's role can be seen as identifying tasks that pose appropriate challenges for students that should result in re-organization of students' ideas and construction of more sophisticated mathematics. From a social point of view, the teacher's role is to see the classroom as a mathematics learning community, and to support it by creating and sustaining the norms and mathematical practices that are taken-as-shared by its members.

Sociological analysis of mathematical classrooms focuses on the concept of taken-as-shared knowledge. As elaborated by previous studies, taken-as-shared knowledge means that, as individuals in a social setting have no direct access to each others' understanding, they achieve a sense that some aspects of the knowledge are shared through verbal and written communication. However, whether this knowledge is actually shared or not cannot be ascertained (Cobb, Yackel, & Wood, 1992).

*Toulmin's Model of Argumentation*

In this study, Toulmin's model of argumentation (1969) was used in order to analyze the teacher's role in the discourse that mainly focused on explanation and justification rather than just answers. The core of the argumentation model includes *data*, *claims* and *warrants*. Data can be described as evidence for the claim and the warrant is the justification that is made in order to connect the data to the claim. Finally, a *backing* may be provided to justify the argument and serves as the validity element of the argumentation. A backing supports why the argument should be accepted by other people as valid mathematical reasoning (Rasmussen & Stephan, 2008).

The concepts used in Toulmin's model of argumentation can better be explained using an example. Imagine that in a fictitious classroom the students are asked to solve the problem of 5 - (-10). A student may make a *claim* that the answer is 15. When challenged by the teacher or the other students, she might provide a data by saying that she reached the answer by finding 5 + 10. At this point, how she rearranged the original expression to reach this new expression may not be clear to some students, so the teacher prompts her to justify her reasoning. In response, she can provide a *warrant* by arguing that subtracting a negative number is actually like adding it up - that's how she reached this answer. This justification may be acceptable to most students but it still involves an implicit assumption that subtracting a negative number somehow translates to addition. Thus, the teacher may solicit a *backing* for how she made this translation. The student might answer this by folding back her imagery of debts and assets, and state that subtracting a negative number is like taking away debt. Taking away debt is a good thing so the net worth goes up. This backing serves to justify the reasoning used in the warrant. In a real classroom conversation identifying these terms may be less straightforward, however they generally fit in the form described. Toulmin's scheme is a useful tool for analyzing inquiry-based environments since the interactions within the classroom community mainly consist of argumentations (Stephan & Rasmussen, 2002).

## Methodology

### *Data Collection*

This study was part of a five-week design research project centered around supporting students' development of integer concepts and operations with theoretical interests on documenting effective planning and classroom teaching practices (Stephan, 2009). Different from many of the other design-based research studies, the aim of the research team in this study was to improve teaching practice rather than to test a theory. The design research team included the researcher, the expert teacher, a co-teacher and two other seventh grade teachers who were teaching the same unit. The research team met three times before the instruction began and every week afterwards. The main focus of these meetings was to evaluate the students' learning and improve the instruction accordingly. The team anticipated the different strategies that students might invent, talk about the imagery and inscriptions that might support students' understanding, and conjectured about the possible topics that might evolve during the enactment of the activities in the classroom. The team also worked through the tasks of the sequence to anticipate how the students might reason and to clarify the intent of the activities.

The study was conducted in a seventh grade classroom in a public middle school in Central Florida in spring 2009. There were twenty students in the classroom including thirteen boys and seven girls. Data sources used in this study included audio- and video-tapes of the classroom sessions, field notes, teacher notes, and a collection of students' artifacts. The data were used to describe the role of the teacher in supporting imagery during interactions with the students. The teacher in this study is considered an expert teacher based on the model of Leinhardt and Smith (1985). According to this model, an expert teacher is defined based on cognitive aspects such as the teachers' knowledge and skills. Experts' knowledge is defined as better integrated, more accessible, and organized in specific ways such as the ideas are connected and the relationship between the ideas can be clearly specified. Expert teachers have a more refined hierarchical structure that includes not only the procedural rules but also the interrelationship of the procedures. They build upon the instructional topics that are introduced in previous lessons and display considerable sophistication of the subject matter during instruction. The teacher analyzed in this study is considered to be an expert teacher primarily based on her cognitive aspects of teaching. Her students have shown consistent growth scores over time, and she has been teaching for more than five years using inquiry-based practices. Although the time in service was not a main requirement, the assumption that the expert teacher had developed effective routines in the teaching and planning practices during this time was an important factor. Another important consideration was the teacher's active involvement in research related with supporting students' thinking during the mathematical instruction. Prior to full time teaching, she had been a professor of mathematics education and specialized in designing instruction to support inquiry-based teaching approaches.

### *Data Analysis*

To find the classroom practices of the expert teacher that contribute to the development of meaningful imagery we conducted our analysis in two phases. In the first phase, we transcribed the classroom discussions over the course of the study. This resulted in more than 160 pages records of classroom discussions. From these transcriptions, we derived three main images that students used to support their reasoning in argumentations. In the second phase, we investigated the role of the teacher in helping students develop those images by analyzing the argumentation logs from the transcripts. We focused on finding instances of when students challenge a claim made by other students, when a previously challenged claim no longer needs justification, and when statements made in claims, warrants, and backings shift position so that they appear as data in Toulmin's model of argumentation. To this end, argumentations of students for different activities were compared within the same day as well as across different days. This method is consistent with the constant comparative analysis since the results were generated and the examples were selected after contrasting different situations (Glaser & Strauss, 1967).

According to Stephan and Rasmussen (2002), these instances signal that a mathematical image or idea has become taken-as-shared in the classroom. Once we determined these instances, we focused on the instructional activities and the actions of the teacher that supported the development of the images that were used in these instances. These activities and the actions were then considered as part of the teacher's practice that supported the development of meaningful imagery and documented in this article.

## Results

In the following sections, we first discuss the imagery that the teacher supported followed by the instructional activities that provided the opportunities to support development of the students' imagery.

### *Situation-specific Imagery: Assets, Debts and Net Worth*

The instructional sequence in this study started with a context that involved determining a person's financial net worth. The focus of the beginning activities was to support an image where assets and debts are quantities that have opposite effect on net worth (Stephan, 2009). The teacher started the sequence with a context where students can use their informal knowledge of owning and owing money for items to construct an image and definition of net worth. In the beginning of the sequence, the teacher asked students to name examples of a famous person's properties that she may *owe* money on or *own*. During the classroom discussion students guessed different things that she may own such as houses, boats, loans as well the things that she may owe money on such as car loans, credit cards and mortgages. The teacher divided the board into two halves and listed the examples for owned items on one left side and the owed ones on the right. She then asked students if anyone knew the names of these categories. One of the students mentioned they are called *debts*. Since no one could come up with the name for the left hand side, she expressed that those are called *assets*. Next, the teacher asked students to define *net worth* and challenged them to describe it in terms of assets and debts. This was followed by an activity that included a net worth statement. Students reviewed the statement and found the words that were familiar and unfamiliar to them (e.g. stocks, mortgages). This activity was the underpinning of the instructional sequence where the students could develop imagery to support that assets and debts are quantities that affect net worth in opposite directions, one positively and one negatively.

In the sessions that followed, the students were asked to create their own net worth statements and find out their total net worth. This was followed by similar activities where students computed the net worth of other people. In the first question which asks the net worth when the assets and debts are $940,000 and $850,000 respectively, most of the students did not have difficulty finding the correct answer. By using Toulmin's model of argumentation we interpret the conversation between the teacher and students in the classroom as an instance of negotiating the idea that a net worth is the difference between total assets and debts. Figure 1 represents the discourse in terms of Toulmin argumentation.

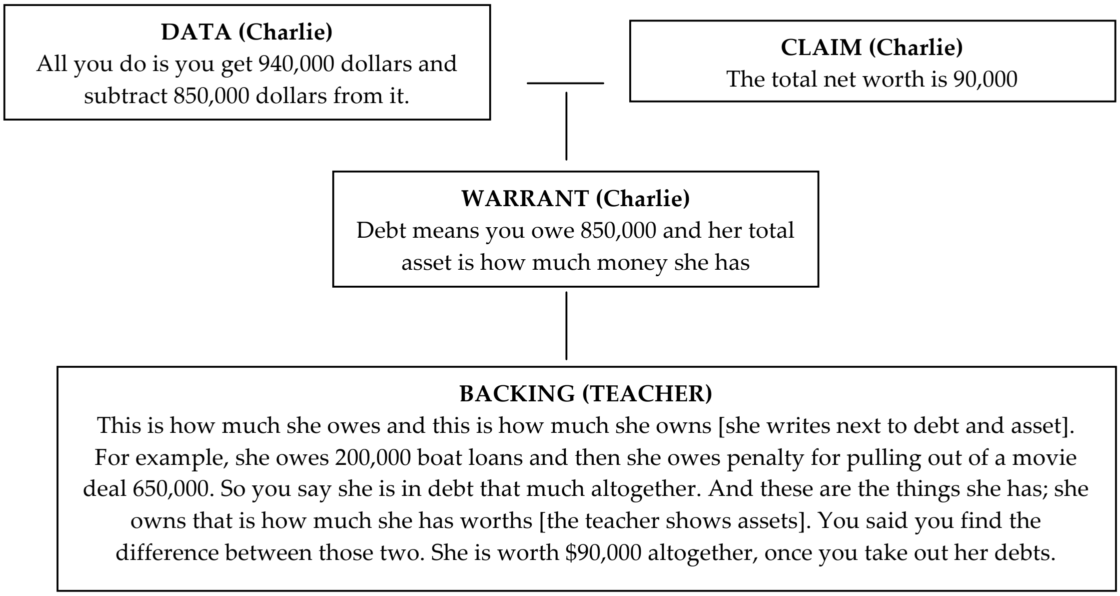


*Figure 1:* Toulmin's representation of an argument for finding the net worth

In this problem, Charlie provided a DATA that he found the answer by computing the difference between the assets and debts. However, the teacher was not satisfied by the answer, since she wanted them to make sense of what those numbers represented for supporting the subsequent activities. Thus, she asked for WARRANT and Charlie explained his answer based on the situation specific imagery. Next, the teacher summarized his reasoning with a BACKING that was based on the situation specific imagery by giving examples from real life such as a boat loan and a penalty. The aim of the teacher in these activities was to ground students' future integer activity in the financial context of net worth and the effect that assets and debts have on it. The activities were designed to support students' activity in this context as conceptualizing asset as something owned, debt as something owed, and net worth as an abstract quantity that you have when debts are taken out from assets. Later class periods indicated that students have taken-as-shared that debts are what you owe and assets are what you own as they used these terms in their DATA in solving net worth problems.

As the instructional sequence moved forward, the teacher started to use + and – signs to replace the words asset and debt on the activity sheets. During this shift in symbolization, students did not question why these symbols were used; it was taken-as-shared that a positive sign signified an asset and a negative sign was a debt. Following that the teacher often did not use the words such as bank account or car loan when asking questions. Here, it is important to note that the teacher began the instructional sequence by using students' words such as own and owe. She then defined the more formal terms asset, debt, and net worth from their contributions and helped students discover the opposite effect of assets and debts on net worth by using examples from the context. Finally, she introduced the mathematical symbolizations to positive and negative numbers. At this point, although the teacher moved away from using informal examples, the students could fold back to their initial images when they needed. For example, in the discussion of a question which asked finding the net worth of Cody when his assets were 325 and debts were 450, Seth offered the explanation that "first of all the debt is how much he owes [he writes next to 450], this is [he writes next to 325] how much he owns. Since he owes more than he owns, so if it is a car whatever he had, and he sold it, he would have 325 dollars".

*Notational Imagery: Flexibly Structuring Space Vertically*

As the instructional sequence moved on, some students had already begun to develop the notion of paying off debts using assets. The teacher wanted to capture these ideas by introducing a vertical number line to provide an inscriptional device for the students to record their reasoning. The teacher preferred to use the vertical number line over the horizontal number line as she believed that the former better capitalized on the students' imagery that the net worth goes *up* by adding assets and *down* by adding debts. During the argumentation of a problem that asks for the total net worth when the total debts are $8,400 and the assets are $8,000 the teacher capitalized on a student's personal notation to introduce the vertical number line (see Figure 2). In the discussion, the students claimed that the person could use his assets to pay off his debts. But they also recognized that he has more debts than assets so he cannot pay off all his debts. To emphasize one of the students' pay off idea, the teacher asked how much he could actually pay off. Students provided DATA that although he uses all his assets he would still have $400 in debt. The claims of the students showed that the idea of assets as positive, debts as negative and net worth as the difference was taken-as-shared.

At this point, to help students visualize their thinking and support their activity in subsequent tasks, the teacher introduced a notational imagery of the vertical number line together with arrows. However, it is important to note that the teacher did not introduce the number line as ready-made but introduced it in a way to capture students' ideas and reasoning. This allowed students to better understand why the tool was needed and how it could be used to develop and communicate their ideas. Introducing tools based on and to facilitate students' reasoning was a commonly employed practice of the teacher.

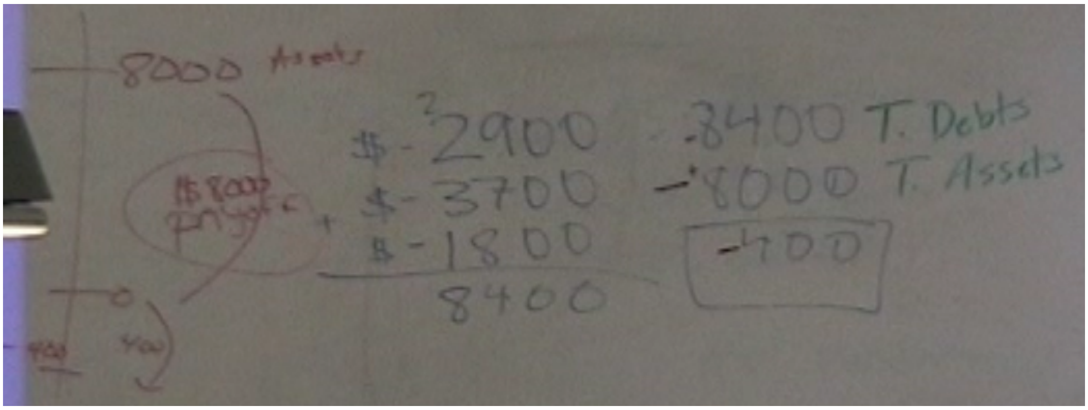


*Figure 2:* Demonstration of pay off idea on the vertical number line

During the solutions of these types of problems the teacher used the vertical number line to help students make sense of computations. Especially at the beginning of the instructional sequence many students tried to find the answers by always subtracting the smaller number from the bigger one, and using the sign of the bigger number. By capitalizing on students' imagery of paying off and left over, the vertical number line became an inscriptional device that made student's imagery visible. In the example above, the teacher capitalized on the idea that the students subtracted one number from the other because they wanted to know what was left over after paying off their debts. For example in this question he has to pay $8400 but he only has $8000. Thus after paying $8000 he went down to zero and he was left with $400 in debt. The pay off image made visible by the number line also supported students' understanding of why and when to subtract. Helping students to make sense of their computations by using imagery of paying off was an important part of the teacher's practice. In the next class, the teacher asked students to find out the new net worth when assets and debts were 325 and 450 respectively.

Students could easily answer the problem with the reasoning that the person has more debts than his assets thus he first needs to get to zero and then down 125. Nobody asked for BACKING or WARRANT. The argumentation indicated that paying off to find a person's net worth was taken-as-shared. Although, sometimes students did not use the number line, it can be concluded from their words that they seemed to develop image of vertical number line. In these activities the number line served as a tool to help students visualize the positive and negative net worth. Using the number line as a computational device was later introduced in the transaction activities.

In the following lesson the teacher gave an activity that supported students' notational imagery but with different reasoning. The aim of the teacher in these activities was to help students to compare two different net worths and find the difference between them. For this reason, the teacher introduced a black and-red number line where the positive parts of the number line were shown in black and the negative parts were shown in red. Defining a new number line in this way built on the financial context as well as on students' reasoning; the teacher also connected this idea with the financial concept of "being in the black" and "being in the red". Here, the teacher first asked if the students knew what it means to be in the black and to be in the red. Some students conjectured that being in the black is associated with having assets and being in the red is associated with having debts. The teacher capitalized on these thoughts and supported students developing an image where black and red colors are associated with positive and negative parts of the vertical number line. Next, the students were given the problems that asked them to order the numbers on the number line such as -$22,000 and -$20,000. This time the teacher asked the students whether they heard the expression "climbing your way out of debt". Another student offered his own expression, "digging yourself into a hole". In this problem students' DATA suggested that they seemed to develop an imagery that as they go down on the number line they go into more debt (get into a deeper hole) and if they go up on the number line they climb out of debt (out of the hole). Although this was the first time that ordering negative numbers emerged as a topic of conversation, students could correctly order the numbers on the line. At the end of the discussion students stated that the numbers that are "more" negative should be below the numbers that are "less" negative. The conversation in the classroom supported that students have taken-as-shared the idea that the higher negative net worths would be further down away from zero.

*Situation-specific To Notational Imagery: Good and Bad Decisions*

Once flexibly structuring space became taken-as-shared, the instructional sequence continued with activities involving transactions with integers. These activities began with tasks such as finding the new net worth given the original net worth and the transaction amount. This was followed by more challenging tasks such as finding the original net worth given the transaction amount and the new net worth. To help in these tasks, the teacher first introduced a task involving good and bad decisions. Through several activities, she asked students to determine whether actions such as adding assets, adding debts, taking away assets, and taking away debts are good or bad decisions. Once the students could easily recognize whether a certain transaction was a good or bad decision, the teacher introduced a symbolization to write these transactions. For example, taking away a debt of 10 was written as –(-10), where the first sign showed the action and the second one whether the quantity was a debt or asset.

During these discussions, one of the students conjectured that +(+) and –(-) were both good decisions but +(-) and –(+) were bad decisions (at this point almost all students wanted to make this conjecture but the teacher named it Stuart's conjecture as he was the first one say it out). Figure 3 shows how the students provided a DATA for this conjecture by arguing that +(+) means adding asset and –(-) means taking away debt both of which are good decisions as they make the net worth go up. The +(-) and –(+), on the other hand, stand for adding debt and taking away asset which are bad decisions as they result in the net worth going down. Since BACKING and WARRANT did not apperar in this argument, it suggests that the images of the context (asset and debt) and good and bad decisions were taken-as-shared by the students at this point in the instruction.

**DATA**

You are making money because you are taking away debt and it is the same if you have both positive signs inside and outside. If you have positive outside and negative inside it is not good, it is also bad if you have negative outside and positive inside.

**CLAIM**

+(+) and –(-) are good decisions
+(-) and –(+) are bad decisions

*Figure 3:* Toulmin's argumentation for good and bad decisions

It is important to note that some students had difficulty in assimilating the meaning of this symbolization so the transition from the good and bad decisions to the corresponding symbols was gradual. For instance, in one question the initial and final net worths were given as $10,000 and $12,000, and the students were asked to find the possible transactions that might create this situation. One student suggested –(-2,000) as a possible answer, but Charlie disagreed. When the teacher asked if he can explain his reasoning, he said "Because you are minusing … never mind I agree. Minusing debt is like she owed $2,000 and then she did not have to pay it so she went up". By folding back to his situation-specific imagery, the student could give meaning to the mathematical symbolization.

Good and bad decision imagery emerged first as situation-specific imagery as students tried to decide which transactions were profitable or not. Students' imagery of good/bad decisions quickly shifted to notational imagery as they wrote symbols to represent the transactions (e.g., +(-30)). To strengthen students' imagery associated with good and bad decisions, as well as the previous images, the teacher engaged students in various activities where they could use these images collectively. In the following classroom the teacher orchestrated the discussion with an activity which involved finding the original net worth given the new net worth and the transaction. The question was finding what the box should contain to satisfy □ - (-30) = 10.

Although many students could solve the problem by using the "backwards" method, some students had difficulty in understanding the solution of the problem that was explained by their classmates. In order to help students that have difficulty, the teacher initiated this with a BACKING by elaborating the explanation of the students to make sure that the explanation could be clearly understood by everyone. She first wanted students to determine whether taking away a debt of 30 is a good or bad decision. Students had no difficult in identifying it as a good decision as this was already taken-as-shared. The teacher then asked students whether this transaction would move the net worth up or down. It was also established that a good decision moves the net worth up. However, because the original net worth was unknown, they had to work backwards starting from 10 and going down to first 0 and then to -20 to complete the total transaction (Figure 4). By going from a positive number to a negative one in two steps, the teacher also emphasized the image of structured space.

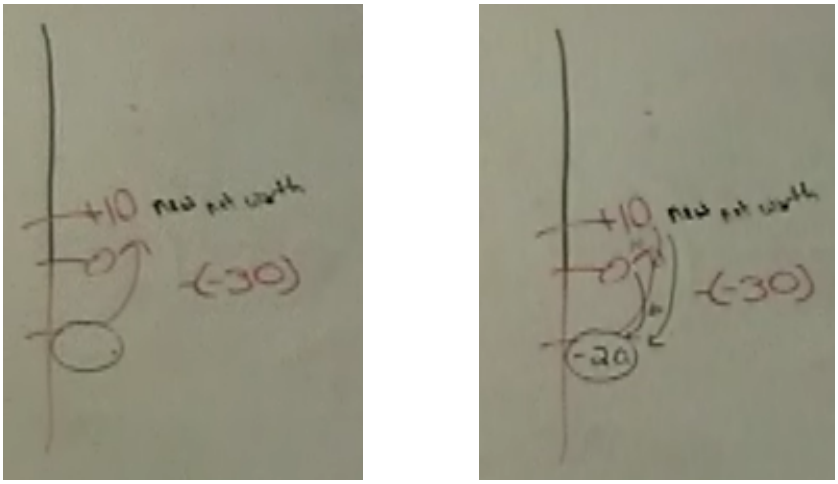


*Figure 4:* Finding the original net worth from the new net worth and transaction

Elaborating students' explanations to make it more accessible to other students by asking questions and presenting them in a more organized manner, and connecting it with the imagery that was already developed constituted an important part of the teacher's practice. This practice was commonly employed to overcome students' misconceptions and difficulties.

During this discussion, the students used the notational imagery with different reasoning. In order to make sense of the number line first they folded back the situation specific imagery of "good and bad decisions". This helped them find the direction that they needed to go on the number line. With the help of the number line they could visualize zero as a breaking point and found the answer by making two jumps. Thus in this argumentation both situation-specific and notational imagery helped students to make sense of the problem.

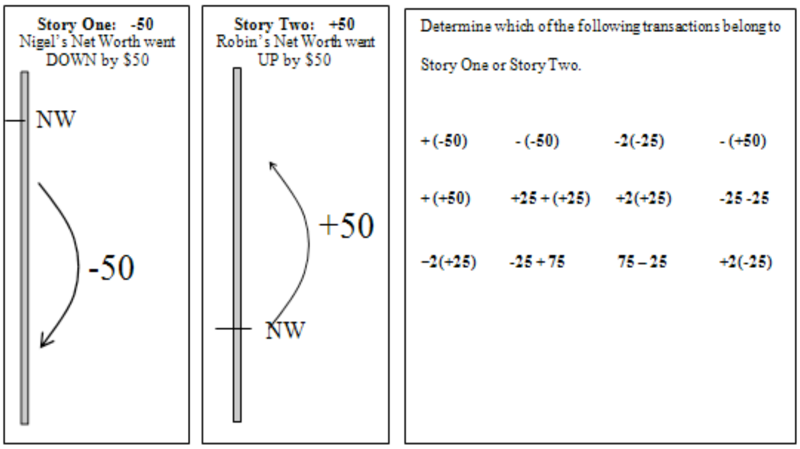


*Figure 5:* Matching transactions that contain only one sign

In order to connect the integer operations with the situation-specific imagery of debts and assets, most activities until this point involved numbers with two signs, i.e. –a + (-b) instead of –a – b. However, most textbook examples contain expressions with one sign and therefore it is important for the students to understand the equivalence of these expressions. To help students make sense of this notation, the teacher prepared activities that involved matching different transactions with numbers such as +50 and -50 within the context of a story as illustrated in Figure 5. These activities helped students recognize the equivalence of expressions with two signs and those with one sign.

## Discussion and Conclusion

In this paper, we attempted to document and demonstrate the role of an expert middle school mathematics teacher in helping students develop meaningful imagery during the instruction of the concepts of integers and integer addition and subtraction. The instructional sequence, which was prepared previously by the teacher, was grounded in the situation specific imagery of assets, debts, and net worth. The instruction, however, did not immediately start from these concepts but it was progressively built on the concepts of owning and owing which were parts of the informal knowledge of the students. The practice of helping students develop imagery based on their informal knowledge and on the previously established images constituted an essential element of the teacher's practice.

As the instruction moved forward, the students started to develop the concept of paying off debts using assets. To capture these ideas and provide a tool for the students the teacher supported the notational imagery by introducing the vertical number line. Using the vertical number line, the students were able to visualize their reasoning and better communicate their ideas during the classroom discussions. The teacher did not introduce the number line as ready-made but brought it up to capture students' ideas and facilitate their reasoning. Making the tool part of the situation-specific imagery and introducing it based on students activities was an important part of the teacher's practice. This allowed students to better understand why the tool is needed and how it can be used to develop and communicate their ideas.

To support students' activity in more advanced questions, the teacher promoted the image of good and bad decisions. To this end, she first asked what types of transactions can be considered good or bad based on their effect on net worth. Once what makes a transaction good or bad was taken-as-shared, the teacher introduced the mathematical symbolization of the given transactions using addition and subtraction symbols. The practice of introducing formal mathematical symbolizations after establishing the supporting imagery was an important part of the teacher's practice.

Research shows that integers, especially negative numbers, create difficulties for students as they try to make sense of them based on their presuppositions about natural numbers and assume that what they know about natural numbers holds for integers. (Gallardo, 2002; Gallardo & Romero, 1999; Peled, Mukhopadhyay & Resnick, 1990). Additionally, the research indicates that students' misconceptions about integers cause difficulties in algebra, for instance in polynomial reduction operations, simplifying equation members, or finding solution when polynomial terms have negative coefficients (Vlasis, 2001). Thus, a good understanding of integers is crucial for students to be successful in mathematics.

Research also emphasizes the importance of supporting students' development of imagery in order to facilitate their mathematical understanding (Pirie & Kieren, 1989, Presmeg, 1992; Thompson, 1992). The teacher has an essential role in supporting imagery with an appropriate instructional design that can help students develop mathematical concepts (McClain & Cobb, 1998). However, there are not many studies that focus on the teachers' role in supporting imagery where they can create meaning for their mathematical activity. The current study addresses this issue by analyzing the teacher's role in supporting imagery during the instructional sequence of integers.

The findings reveal that various situation-specific and notational images that students developed during the instruction of the integers helped them understand and solve the mathematical problems meaningfully. They often used these images in explaining their reasoning in classroom and small group discussions. The students used the initial images in different types of integer problems including subtraction and different signs. Thus, the supported images were not only effective for addition but also for subtraction as well. This was an essential result since many of the models that were described in the literature had some deficiencies in supporting the subtraction especially when the signs were different (Lytle, 1994).

This study demonstrates that imagery can be supported by first designing a learning trajectory which includes a supportive context, tools to be used, notations to be developed, and learning goals to be achieved. This initial plan can be refined and continually updated during an inquiry-based instruction within a design research environment that includes the teacher, other teachers teaching similar topics, and researchers. As such, while this study was conducted for integers, the methodology of the teacher can be applied to support imagery in different topics and areas.

Teaching and learning to teach in an inquiry environment is challenging as reported by the earlier studies on teacher education. These studies highlight that pre-service teachers struggle with the feeling of losing control, following students' interest, and asking the right questions to bring forth stimulating discussions (Hayes, 2002). Therefore, the results of the current study can be used in pre-service teacher's courses as well as in professional development to demonstrate the use of imagery to support students' mathematical understanding in an inquiry environment. The researchers might also use the instructional sequence described in this study to compare it with the traditional methods used for teaching integers. Most importantly, the effects of supporting imagery on students' learning in subsequent years might be investigated, and the achievement of students who learned integers with the method described in this article and those who learned traditionally can be compared.

Our study leaves some questions unanswered that can be addressed by future research. First, the instructional sequence that we used was based on debts and assets. Although early research indicates that students who were given story problems involving debts and assets performed better than those who only solved formal problems (Mukhopadhyay, Resnick, & Schauble, 1990), a rigorous comparison of this model with other neutralization models is missing. Secondly, although the instructional sequence involved activities to help students make the transition to using single sign such as 5 – 10 instead of 5 – (+10), this notation remained less clear to some students. It could be heard that they were not sure whether the sign represented a negative or a take away operation. Therefore, the presented instructional sequence can be revised to help students solve this dilemma and understand that even though negative and take away are conceptually different they yield the same result. Finally, the instructional sequence was primarily focused on addition and subtraction with a slight transition to multiplication toward the end. A useful extension would be to support the instructional sequence with more activities that are focused on multiplication and division.

## References

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher*, 30(9), 26-31.

Cobb, P., & McClain K. (2001). An approach for supporting teachers' learning in social context. In F. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 27-231). Netherlands: Kluwer Academic Publishers.

Confrey, J. (1990): What constructivism implies for teaching. In: R. B. Davis, C. A. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 107–122). Reston, VA: National Council of Teachers of Mathematics.

Dirks, M (1984). The integer abacus. *Arithmetic Teacher,* 31(7), 50-54.

diSessa, A. A., & Sherin, B. L. (2000). Meta-representation: An introduction. *Journal of Mathematical Behavior*, *19*(4), 385–398.

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory*. Chicago: Aldine.

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, Netherlands: CD-Press.

Gallardo, A., & Romero, M. (1999). Identification of difficulties in addition and subtraction of integers in the number line. In F. Hitt, & M. Santos (Eds.), *Proceedings of the Twenty-first*

*International Conference for the Psychology of Mathematics Education* (Vol. I. pp. 275–282). North American Chapter, Mexico.

Gallardo, A. (2002). The extension of the natural-number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics*, 49, 171–192.

Hayes, R. (1999). Teaching Negative Number Using Integer Tiles, 22nd Annual Conference of the Mathematics Education Research Group of Australasia (MERGA), University of Adelaide, Adelaide, SA.

Hayes, M. (2002). Elementary preservice teachers' struggles to define inquiry-based science teaching. Journal of Science Teacher Education, 13, 147-165.

Janvier, C. (1985). *Comparison of models aimed at teaching signed integers*. Proceedings of the Nineth Meeting of the PME. State University of Utrecht, The. Netherlands, 135-140.

Kinach, B. M. (2002). A cognitive strategy for developing pedagogical content knowledge in the secondary mathematics methods course: toward a model of effective practice. Teaching and Teacher Education, 18(1), 51-71.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work. *Educational Psychologist*, *41*(2), 75-86.

Linchevski, L., & Williams, J. D. (1999). Using intuition from everyday life in "filling" the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics*, *39*, 131–147.

Leinhardt, G., & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247-271.

Lytle, P. (1994). Investigation of a Model Based on the Neutralization of Opposites to Teach Integer Addition and Subtraction. In *Proceedings of the 18th International Group for the Psychology of Mathematics Education*, Vol. 3, pp. 192-199) Concordia University, West Montreal, Canada.

Maccini, P., & Ruhl, K. L. (2000). Effects of a graduated instructional sequence on the algebraic subtraction of integers by secondary students with learning disabilities. Education and Treatment of Children, 23, 465–489.

Mukhopadhyay, S., Resnick, L. B., & Schauble, L. (1990). Social sense-making in mathematics; Children's ideas of negative numbers. In G. Booker, P. Cobb, & T. N. de Mendicuti (Eds.), *Proceedings of the 14th international conference for the Psychology in Mathematics Education,* Vol. 3, pp. 281-288, Oaxtepec, Mexico: Conference Committee.

McClain, K., & Cobb, P. (1998). The role of imagery and discourse in supporting students' mathematical development. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 56–81). New York: Cambridge University Press.

Pinto, M. M. F, & Tall, D. O. (1999). Student constructions of formal theory: giving and extracting meaning. In O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of PME, Haifa, Israel, 4, 65–73.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics*, *9*(3), 7–11.

Presmeg, N.C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics,* 23, 595–610.

Rasmussen, C., & Stephan, M. (2008). A Methodology for Documenting Collective Activity. In A. E. Kelly, R. A. Lesh & J. Y. Baek (Eds.), *Handbook of design research in methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 195-215). New York and London: Routledge.

Schwartz, D. L., & Martin, T. (2004). Inventing to prepare for future learning: The hidden efficiency of encouraging original student production in statistics instruction. *Cognition* *and Instruction, 22*(2), 129-184.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior*, *21*, 459–490.

Stephan, M. (2003). Reconceptualizing Linear Measurement Studies: The Development of Three Monograph Themes. In M. Stephan, J. S. Bowers, P. Cobb & K. P. E. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (*Journal for Research in Mathematics Education*, Monograph number 12, pp. 17-35). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M. (2009). What are you worth*? Mathematics Teaching in Middle School, 15(1),* 16-23.

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education*, *19*(2), 115-133.

Thompson, P. W. (1992). Notations, conventions, and constraints: Contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education*, *23*(2), 123-147.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (267–285). Mahwah, NJ: Erlbaum.

Vlassis, J. (2001). Solving equations with negatives or crossing the formalizing gap. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the twenty-fifth international conference for the psychology of mathematics education* (Vol. 4, pp. 375–382). Utrecht, Netherlands.